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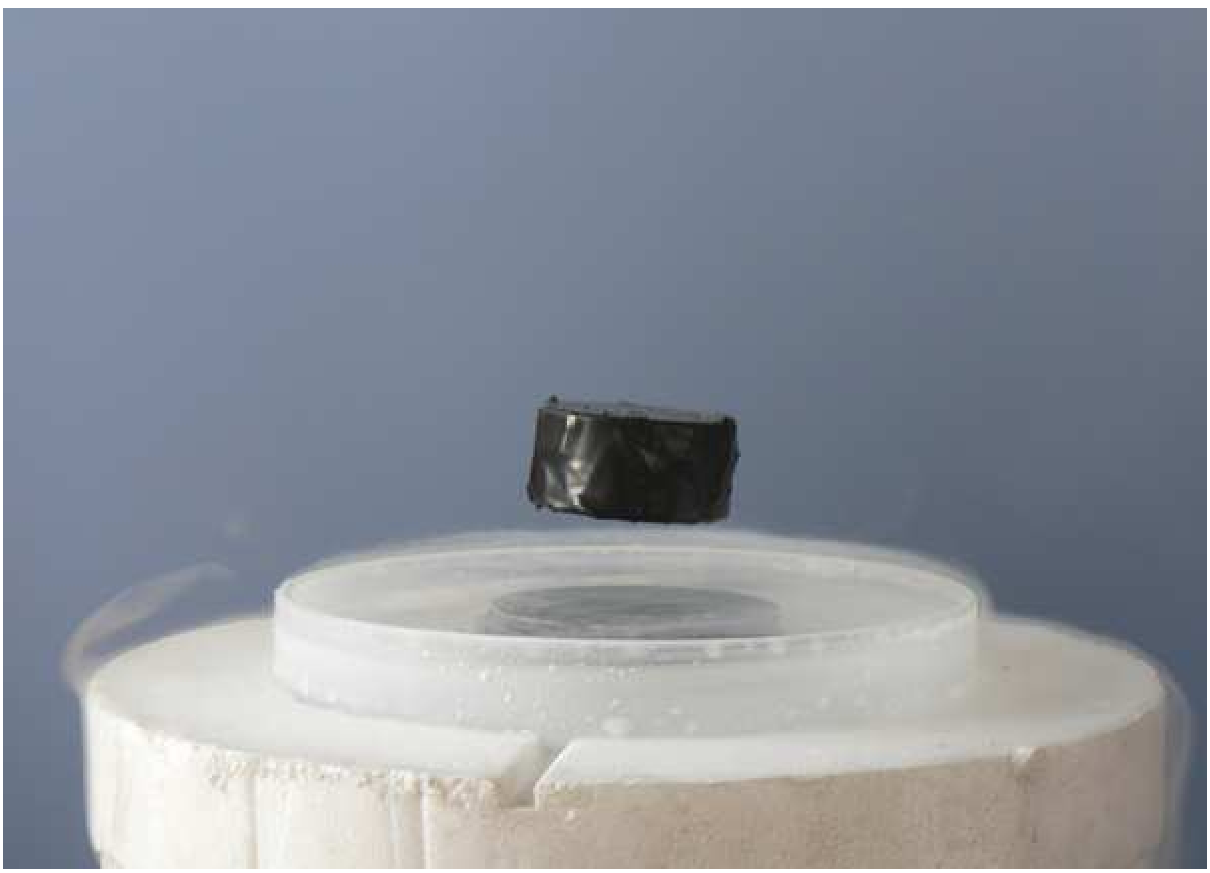

International Association of Mathematical Physics News Bulletin, April 2011

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http://www.iamp.org and e-mail: bulletin@iamp.org

Cover photo: A magnet levitating above a superconductor, cooled with liquid nitrogen. Superconductivity was discovered 100 years ago, on April 8, 1911 by Heike Kamerlinh Onnes at the University of Leiden.

Think of the best ones

by Pavel Exner (IAMP President)

The Association life brings opportunities to see our activity from different angles and time perspectives. This issue provides a good illustration of this claim. On one hand it brings reflections on BCS theory, which was a source of inspiration over several decades, on the other hand it reports about the recent events concerning the Erwin Schrödinger Institute in Vienna. Here the IAMP and many of its members individually joined the international protest against a scandalous political decision and I hope we contributed in this way to saving one of our oldest Associate Members.

On the background of such events, large and small, the Association life proceeds according to its basic rhythm. One of the most important things in the pre-congress year is the prize nominations, and I want to ask you to notice the calls published in this issue.

True, there are other prizes, and we are happy when our members are distinguished in sister communities. Some time ago I praised in these pages Stanislav Smirnov and Cédric Villani as fresh Fields laureates, and one can also mention Srinivasa Varadhan's Abel Prize in 2007, Shing-Tung Yau's Wolf Prize last year, and others. Nevertheless, it is our own prizes to which we have to pay most attention.

The most important among them is the *Henri Poincaré Prize* which exists due to the generous support of the *Daniel Iagolnitzer Foundation*. It is the oldest among the mathematical physics prizes; next year in Aalborg it will be awarded for the sixth time. Looking at the prize description and the list of laureates at

http://www.iamp.org/page.php?page=page_prize_poincare

it is clear that its aim is to distinguish the biggest achievements and the brightest promises in our field.

In contrast to the Poincaré Prize, which has already established its renown, the two other prizes are still new and were awarded in Prague for the first time. What is important is that both target young people, irrespective of their different descriptions and rules. The *IAMP Early Career Award* is our Association's own prize established by the previous Executive Committee, while the *IUPAP Young Scientist Prize* was established by the International Union for Pure and Applied Physics, Commission for Mathematical Physics, with which the IAMP traditionally closely collaborates.

Identifying colleagues and results we can be proud of is one of the important IAMP tasks. I invite all of you to think who should be distinguished when we meet next August in Aalborg, and to make nominations in time so that the prize committees can evaluate the candidate merits properly.

Superconductivity, BCS theory and mathematical physics

by Jean-Bernard Bru & Walter de Siqueira Pedra
(Bilbao, Spain & Mainz, Germany)

Jean-Bernard Bru is an Ikerbasque Research Professor of the Department of Mathematics at the Faculty of Science and Technology at the UPV-EHU in Leioa (Spain) who recieved his Ph.D. in mathematical physics in 1999 at the University of Aix-Marseille II. Walter de Siqueira Pedra holds a "Juniorprofessor" position for Analysis at the Institute of Mathematics at the Johannes Gutenberg University in Mainz (Germany). He finished his Ph.D. in mathematical physics in 2005 at the Max Planck Institute for Mathematics in the Sciences and the University of Leipzig. Both are researchers in the domain of quantum statistical mechanics in relation to operator algebras, probability theory, convex analysis and functional analysis. Jean-Bernard Bru's first research works were about Bose-Einstein condensation and the Bogoliubov theory of superfluidity, and Walter de Siqueira Pedra started his scientific career with applications of renormalization group methods to the rigorous analysis of KMS states of interacting fermions in crystals. Both researchers have worked together for some years on rigorous approaches to superconductivity and, more generally, on the mathematical foundations of quantum statistical mechanics.

2011 closes a century of superconductivity, but its study has continued to intensify until nowadays. In the year 1957, J. Bardeen, L. N. Cooper, and J. R. Schrieffer proposed the first convincing microscopic theory of superconductivity, the so–called BCS theory [1, 2, 3], for which the Nobel Prize in Physics was awarded in 1972. Since the discovery of mercury superconductivity below 4.2 K in 1911 by the Dutch physicist Onnes, a significant number of superconducting materials has been found. This includes ordinary metals, like lead, aluminum, zinc, or platinum, but also magnetic materials, heavy–fermion systems, organic compounds, and ceramic compounds (cuprates and others), which yield superconductors at temperatures above 100 K. The discovery of superconductivity at high critical temperatures (i.e., above 10 K) in 1986 by K. Müller and J. Bednorz, who received the Nobel Prize in Physics in 1987, represents an important experimental breakthrough and the understanding of its microscopic origin is still one of the most important challenges to physics.

Meantime, theoretical physics took a large step forward in terms of abstraction since the beginning of the 20th century. This mainly corresponds to the birth of a geometric theory of gravitation, that is, the general theory of relativity achieved by A. Einstein in 1916, and to the emergence of quantum mechanics, first advocated by N. Bohr and other well–known physicists. This development was supported by experimental facts, like helium superfluidity or superconductivity, whose description goes beyond the scope of classical theories. This led to an increasing of mathematical sophistication in the description of physical systems and mathematical physics naturally emerged in the fifties and sixties as a field of knowledge. One could argue that Isaac Newton, for instance, may retrospectively be considered as a mathematical physicist, but the first peer–reviewed journals devoted to this branch, as such, only appeared in the sixties (J. Math. Phys. 1960, Commun. Math. Phys. 1965). Textbooks (see, e.g., [4, 5]) and the IAMP appeared about a decade later.

Being a major problem in Physics, superconductivity has interpenetrated mathematical physics from the beginning, in particular through the BCS theory [1, 2, 3]. Indeed, [1] provides an interpretation of this physical phenomenon in terms of formation of pairs of electrons, since then known as Cooper pairs. [2, 3] give the theoretical foundations of such pair formation via the so–called BCS Hamiltonian. Using a non–rigorous treatment of the BCS Hamiltonian, the BCS theory was able to easily explain many aspects of the phenomenology of conventional (type I) superconductors. The first successful attempts to mathematically clarify this theory were performed by N.N. Bogoliubov [6] in 1960 and by R. Haag in 1962 [7]. N.N. Bogoliubov and R. Haag were, respectively, a mathematician and a (theoretical) physicist who both committed themselves to mathematical physics.

Indeed, after having worked in mathematics, N.N. Bogoliubov became famous among physicists for his microscopic theory of helium superfluidity presented in 1947. Ten years earlier than Cooper's proposal of electron pairs in superconductors, his theory yields a Landau–type excitation spectrum with formation of quasi-particles made of boson pairs with opposite momenta; see [8] for a detailed review. In [6] he solved the problem of the ground state of the BCS Hamiltonian and for this reason the BCS theory is sometimes referred as the *BCS-Bogoliubov theory*. This result was the starting point of a series of rigorous studies on the BCS theory, in particular on thermodynamic properties of the BCS Hamiltonian.

N.N. Bogoliubov pointed out the strong analogy between superconductivity and superfluidity (see the introduction of [9]) and, indeed, most of theoretical approaches, including the BCS theory as originally presented in [1, 2, 3], are based on the so–called Bogoliubov approximation, used for the first time in Bogoliubov's 1947 theory of superfluidity. Roughly speaking, this approximation consists in replacing specific operators appearing in the Hamiltonian of a given physical system by constants which are determined as solutions of some self–consistency equation, the so–called gap equation in the BCS theory, or of some associated variational problem. More than two decades later, this idea gave origin to a systematic method used in mathematical physics, the Approximating Hamiltonian Method introduced by N.N. Bogoliubov Jr. in 1966 [10, 11] and by J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov, and N.S. Tonchev in the seventies and eighties [12, 13, 14], see also [15, Sect. 2.9 & 10.2] for a short review. This mathematical approach corresponds – in the class of Hamiltonians for which it applies – to a rigorous method of proving the exactness of the Bogoliubov approximation in the thermodynamic limit, on the level of the pressure, provided such an approximation is done in an appropriated manner. In particular, this method gives insight into how certain parts of the Hamiltonian should be replaced by constants.

Some aspects of the BCS theory have a strong connection to the theory of spin systems, initiated with the celebrated Ising model, which describes ferromagnetism in statistical mechanics and was solved in 1925 (in one dimension) by E. Ising. Thus, the theory of superconductivity also stimulated the mathematical development of this rather old subject. Considered as a mathematical formulation of a physical theory, the theory of quantum spin systems is one of the most elegant branches of quantum statistical mechanics (see, e.g., [5, 16, 17]), and the quantum spin representation of fermion systems (Jordan–Wigner transformation) makes possible the use of its formalism in the study of BCS theory. In the Sixties, W. Thirring and A. Wehrl [18, 19] used such a representation to solve the strong coupling BCS model, a simplified version of the original BCS model, which, however, displays most of the basic properties of real conventional type I superconductors. [20, 21] are examples of more recent results involving the strong coupling BCS model. Then, using again the quantum spin representation, the thermodynamics of the (usual) BCS Hamiltonian was rigorously analyzed, on the level of the pressure, in the eighties [22, 23]. (See also the previous work of I.A. Bernadskii and R.A. Minlos [24] done in 1972.) This approach led to further interesting mathematical results on the thermodynamics of certain quantum spin systems related to the BCS model, see, e.g., [25] and references therein. However, one crucial drawback of the quantum spin representation of fermions is that it generally breaks the translation invariance of fermion Hamiltonians like the BCS model, i.e., the new spin Hamiltonian is no longer translation invariant.

Observe that the results of W. Thirring and A. Wehrl go beyond thermodynamic results on the pressure alone. Indeed, the simplified model they used allows a detailed description in the thermodynamic limit of its time evolution [18] as well as of its correlation functions [19]. These works, in particular [19], make an explicit use of R. Haag's elegant treatment [7] of the BCS model done in 1962 in relation with the breakdown of the $U(1)$–gauge symmetry in presence of superconductivity. In fact, [7] can be seen as a pivotal work in mathematical physics, for various reasons, inspiring many mathematical studies of the BCS model, as, for instance, [18, 19, 26, 27] in the Sixties. See also [28, p. 28] and the corresponding references therein.

More precisely, the Bogoliubov approximation corresponds to replacing space averages

$$\bar{Q}_\Lambda = \frac{1}{|\Lambda|} \int_\Lambda Q\,(x)\,\mathrm{d}x$$

of operators $Q\,(x)$ within a box Λ of volume $|\Lambda|$ by a complex number $c \in \mathbb{C}$. Applied to the BCS Hamiltonian H^{BCS} it yields (approximating) Hamiltonians $H^{BCS}\,(c)$, which are no longer $U(1)$–gauge invariant, unless $c = 0$. The complex number c must solve the temperature–dependent self–consistency equation (or gap equation) of the BCS theory. R. Haag found a very simple and elegant justification for such replacements: He did not see H^{BCS} as an operator acting on the Fock space but, more generally, as a polynomial in the creation and annihilation operators which are seen as generators of a CAR[1] C^*–algebra $\mathcal{U}$. The thermodynamic phases are then described by states on $\mathcal{U}$, see, e.g., [15, Chap. 1]. He established that $\bar{Q}_\Lambda$ must converge, in any representation of $\mathcal{U}$, to

[1] Canonical Anti–commutation Relations

an operator $\bar{Q}$ commuting with all elements of $\mathcal{U}$, as the volume diverges ($|\Lambda| \to \infty$). Within irreducible representations of $\mathcal{U}$, the limit operator $\bar{Q}$ must then be a complex number $c \in \mathbb{C}$. This number is zero in the usual Fock space representation, whereas the existence of superconductivity is related to non–zero order parameters $c \neq 0$ in the BCS theory. Among other things, this shows the necessity of leaving the usual Fock space representation of the model to go to a representation–free formulation of thermodynamic phases; see also [26] which refines R. Haag's study [7] and [28, Sect. 1.f] for a concise review.

As explained above, the choice of the complex number $c \in \mathbb{C}$ is determined by the gap equation, which is nothing other than the Euler–Lagrange equation coming from the maximization of the c–dependent pressure (or ground state energy) computed from $H^{BCS}(c)$; see, e.g., [15, Sect. 2.7]. As the BCS Hamiltonian H^{BCS} is $U(1)$–gauge invariant, the gap equation only depends on $|c|$ and a superconducting phase means, in this context, the existence of $r > 0$ such that $c = re^{i\varphi}$ is solution of the gap equation for all $\varphi \in [0, 2\pi)$. By [7, 19, 26], for each $\varphi \in [0, 2\pi)$, the non–zero complex number $c = re^{i\varphi}$ labels an irreducible representation of $\mathcal{U}$ with a non–$U(1)$ invariant vacuum (cyclic vector of the corresponding representation). An example of an application of such an approach to a more recent problem is given, for instance, in [29, Sect. 6.2]. Haag's result [7] and its extensions by other authors like [19, 26] put, thus, in very clear mathematical terms the structure of the $U(1)$–symmetry breaking in the context of BCS theory. Note that the breakdown of the $U(1)$–gauge symmetry is also related to the so–called off–diagonal long range order, a property proposed by the physicist C.N. Yang [30] in the same year (1962) to define superconducting phases, see, e.g., [15, Sect. 2.8].

In fact, R. Haag was not originally working on statistical mechanics, but, as he said himself [31, p. 282], "he was infected" by the BCS model of superconductivity [1, 2, 3] and Bogoliubov's result [6] on its ground state. His view point on this subject had also been influenced by his previous works and knowledge on quantum field theory. An important ingredient was his discovery, during a conference in the fifties, of I.E. Segal's mathematical results on C^*–algebras, also obtained, in parallel, by the Soviet mathematicians I.M. Gel'fand and M.A. Naimark. I.E. Segal proposed in this conference to leave the Hilbert space approach to consider quantum observables as elements of an involutive Banach algebras, now known as C^*–algebra, see [31, p. 274]. One important result of the theory of C^*–algebras, performed in the forties, is the so–called GNS (Gel'fand-Naimark-Segal) representation of states, which permits a natural relation between the new algebraic formulation and the usual Hilbert space based formulation of Quantum Mechanics to be established. The GNS representation has also led to very important applications of the Tomita-Takesaki theory, developed in 1970, to quantum field theory and statistical mechanics. These developments mark the beginning of the algebraic approach to quantum field theory and statistical mechanics and, through [7] and subsequent works, the BCS model strongly contributed to this conceptual line, see, e.g., [28].

The algebraic formulation turned out to be extremely important and fruitful for the mathematical foundations of quantum statistical mechanics and have been an important branch of research during decades, at least until the eighties with lots of works, mainly on quantum spin systems, see [5, 16, 17]. After that, the research activity on this subject has decreased, but the last years have seen a rejuvenation of interest in such questions. For instance, H. Araki, one of the researchers who most contributed to quantum statistical mechanics in its algebraic setting, together with H. Moriya have recent results on the algebraic formulation of Fermi systems; see, e.g., [32] and references therein. In 2010 we used the algebraic setting to analyze the thermodynamic impact of the Coulomb repulsion on s–wave superconductors via a rigorous study of equilibrium and ground states of the strong coupling BCS–Hubbard Hamiltonian; see [29]. In [33] we use the algebraic approach in a mathematical study on the emergence of electron–hole asymmetry, as it is observed in superconducting cuprates. The Bogoliubov approximation is again a main heuristic point in [29, 33]. Other recent results go in such a direction. See, e.g., [34, 35].

[29, 33] were initially motivated by the BCS theory and, again, led to new mathematical results on Fermi systems: A recent work [15] gives a complete thermodynamic description of fermions or quantum spin systems in the lattice with long range interactions. In particular, following the spirit of Haag's original work [7], it establishes the validity of the Bogoliubov approximation on the level of states and links the Approximating Hamiltonian Method [12, 13, 14] to the algebraic formulation of the BCS model initiated by R. Haag.

The BCS functional, originally derived from the BCS theory [2, 3] by Leggett in 1980 [36], also motivated other recent mathematical works, not directly related to the algebraic formulation of quantum statistical mechanics, see, e.g., [37, 38] and the references therein. In [37, 38] (and related papers), a detailed rigorous analysis of the gap equation, i.e., the Euler–Lagrange equation associated with the minimization of the BCS functional, is performed and yields some necessary and sufficient conditions to obtain a superconducting phase.

Superconductivity and related topics like the theory of Fermi liquids also inspired very sophisticated analytical results using, rigorously, the renormalization group and constructive methods of quantum field theory. See, for instance, [39] and the series of papers [40].

We would like to conclude by observing that the BCS theory has only been able to explain conventional type I superconductors. A general theory of superconductivity is still a subject of debate, especially for high–temperature superconductors. One phenomenological approach applied to type II superconductors uses the celebrated Ginzburg–Landau theory of superconductivity proposed in 1950, which has also stimulated many recent mathematical results. For an account of this mathematical subject, see, e.g., the monographs [41, 42]. The BCS theory has, however, inspired many mathematical physicists during decades and has revealed important mathematical structures from which we can now better understand various physical situations. As superconductors are still far from being fully understood one century later, it will be so for still a while.

This short review of mathematical results concerning the BCS theory is not exhaustive, and we apologize for those relevant authors and results which were not mentioned.

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Mayer expansions

by David Brydges (Vancouver, Canada)

David Brydges received his Ph.D. in mathematics in 1976 from the University of Michigan (thesis adviser Paul Federbush). He is a leading expert and the author of several fundamental results in the Theory of Cluster Expansions. His lectures: A Short Course on Cluster Expansions, Les Houches 1984, are still one of the most popular sources for many of those who like to enter in this topic. David Brydges is a former president of the IAMP (2003–2005). He holds a Canada Research Chair in the Probability Group at the University of British Columbia. He is well-known for many important results in statistical mechanics and quantum field theory.

Probably all of us can name a few theorems that seemed so magical, when first we met them, that they are a kind of never forgotten first love. In my case there are the two theorems of Joseph E. Mayer on the graphical expansions for the pressure and Helmholz free energy of an imperfect gas. I think they are the first case of expansions labeled by graphs, and graphical expansions have become one of the great themes in mathematical physics. They actually converge, which is surprising for graphical expansions, because there tend to be too many graphs, but there is much more still to be understood about their region of convergence. This article is about these expansions in the context of the hard-sphere system, where they become a very sophisticated exploration of inclusion/exclusion. There are many other reasons to study these expansions, see for example [9].

First, recall the multinomial expansion,

$$\prod_{a\in A}(1+y_a) = \sum_{B\subset A} y^B, \qquad y^B = \prod_{a\in B} y_a. \tag{1}$$

Now consider a gas of hard spheres. Suppose Λ is a huge d-dimensional torus. A configuration of n particles in Λ, which are labeled $1, 2, \dots, n$, is represented by a point $x = (x_1, \dots, x_n) \in \Lambda^n$. If our particles are hard spheres of radius $1/2$ then we must eliminate configurations where any pair of particles is separated by distance less than 1. Therefore, for a pair of labels $i < j$ define a function $f_{ij} = f_{ij}(x)$ on Λ^n by $f_{ij} = 0$ when particles labeled i and j do not overlap and $f_{ij} = 1$ if they do overlap. Then

$$\prod_{1\le i<j\le N} (1 - f_{ij}) \tag{2}$$

is a function defined on Λ^n which is zero if any pair of particles overlaps and otherwise is one. The integral

$$Z_n(\Lambda) = \int_{\Lambda^n} \prod_{1\le i<j\le n} (1 - f_{ij})\, d^n x \tag{3}$$

is the volume of admissible configurations of hard spheres. With $1/n!$ in front it is called the *canonical partition function*.

I think Ursell [11] was the first person to approximate $Z_n(\Lambda)$ by using (1) to expand out the product in (2). For this application of (1) the set A consists of all pairs $i < j$. Think of this as the set of all edges of the complete graph on vertices $1, 2, \ldots, n$. Then $B \subset A$ is the edge set of a graph on the same vertices, so we can rewrite everything in terms of graphs. A graph G is a pair (V, E) where V is the set of vertices and E is the set of edges. With $V = \{1, 2, \ldots, n\}$, Ursell's expansion is

$$Z_n(\Lambda) = \sum_G w(G), \qquad w(G) = \int_{\Lambda^V} (-f)^E d^V x, \tag{4}$$

but expressing it in terms of graphs is Mayer's idea. The papers of J.E. Mayer, starting in 1936, are summarised in his book [6] with Maria Goeppert Mayer. He was a pioneer and his derivations twist and turn, partly because they are based on the canonical partition function. Major simplifications were achieved in work of Uhlenback and Ford [10]. Their starting point is the grand *canonical partition function*, which is the following series in powers of a parameter z,

$$Z(\Lambda, z) = \sum_{n \geq 0} \frac{z^n}{n!} Z_n(\Lambda), \qquad Z_0(\Lambda) = 1. \tag{5}$$

By inserting (4) we find that the grand canonical partition function is the sum over all graphs on $V = \{1, 2, \ldots, n\}$ with $n = 0, 1, \ldots,$

$$Z(\Lambda, z) = \sum_G \frac{z^n}{n!} w(G), \qquad n = |V|. \tag{6}$$

By definition the $n = 0$ term equals one. Mayer's first theorem is the wonderful assertion that $\log Z(\Lambda, z)$ is given by restricting the sum to connected graphs,

$$\log Z(\Lambda, z) = \sum_{G \text{ connected}} \frac{z^n}{n!} w(G). \tag{7}$$

This immediately gives an expansion for the pressure $P = P(\Lambda, z)$ because

$$P(\Lambda, z) = \frac{1}{|\Lambda|} \log Z(\Lambda, z), \tag{8}$$

where $|\Lambda|$ denotes the volume of Λ. In the standard version of this formula, P is replaced by βP, where β is the inverse temperature, but for the hard sphere system, we do not lose useful information by setting $\beta = 1$. The resulting expansion for the pressure is called the *Mayer Expansion*.

Mayer's second theorem is the graphical formula for the coefficients in the virial expansion, which is the expansion for the pressure in powers of the (expected) density ρ. This is given by $\rho = (z\partial P/\partial z)$. Since P is a power series in z, ρ is a power series in

z, and finding the virial expansion involves eliminating z between these two series. It is very surprising that there is a simple outcome to all orders. It is easiest to appreciate his result if we take a little detour. Replace z by an equivalent parameter ν by writing $z = e^\nu$ and define $F = F(\Lambda, \rho)$ by the Legendre transform,

$$F(\Lambda, \rho) = \sup_\nu \big(\nu\rho - P(\Lambda, e^\nu)\big). \tag{9}$$

The standard definition has βF in place of F. The finite volume pressure is a strictly convex function of ν, so F exists and is a strictly convex function of ρ, but with values in the extended reals, $\mathbb{R} \sqcup \{+\infty\}$. What is the expansion for F in powers of ρ? The first reaction is that it does not have one because for the ideal gas $P = e^\nu$ and then from (9) we obtain $F_{\text{ideal}} = \rho \log \rho - \rho$ for $\rho > 0$ and $F_{\text{ideal}} = \infty$ if $\rho < 0$. However,

$$F = F_{\text{ideal}} - |\Lambda|^{-1} \sum_{G \in \text{irreducibles}} \frac{\rho^n}{n!} w(G), \tag{10}$$

where *irreducibles* denotes the set of connected graphs with at least two vertices which cannot be disconnected by the removal of a vertex. Thus the Legendre transform implements restricting the sum to graphs with a higher level of connectedness! The general connection between Legendre transforms and higher order connectedness was made by De Dominicis [3]. From thermodynamics one finds that $F(\Lambda, \rho)$ is the Helmholz free energy per unit volume. In statistical mechanics the Helmholz free energy is $-\log(Z_n/n!)$, but one has to take the infinite volume limit for this to coincide with the Legendre transform definition.

The expansion (10) for F is equivalent to Mayer's virial expansion. To see this, write P in terms of F by using the duality of the Legendre transform, $P = \sup_\rho(\nu\rho - F)$. The supremum is achieved when $\nu = \partial F/\partial\rho$. Therefore,

$$P = \frac{\partial F}{\partial \rho}\,\rho - F = \rho - |\Lambda|^{-1} \sum_{G \in \text{irreducibles}} \frac{(1-n)}{n!} \rho^n w(G). \tag{11}$$

where (10) was used in the second equality. Mayer defines β_n to be the sum over irreducibles on $n+1$ vertices with one vertex distinguished and held fixed and then this expansion is his formula $\rho - \sum_{n \geq 1} n/(n+1)\beta_n \rho^{n+1}$.

At first sight it is surprising that the Mayer expansion converges because there are $2^{O(n^2)}$ connected graphs on n vertices which overwhelms the $1/n!$ in (7). However there are cancellations between graphs at the same order z^n. J. Groeneveld [4] proved that the Mayer expansion converges for $|z|$ sufficiently small. In [7], O. Penrose gave a different proof based on an extraordinary inequality that says that the sum over all connected graphs is less than the same sum restricted to tree graphs!

$$\left| \sum_{G \in \text{connected graphs, } n \text{ vertices}} w(G) \right| \leq \left| \sum_{G \in \text{trees, } n \text{ vertices}} w(T) \right|. \tag{12}$$

The $1/n!$ in (7) is not in the least bit scared of n^{n-2} tree graphs.

Multiplying the density ρ by the volume of the hard sphere gives the proportion of volume occupied by the spheres. This is called the packing fraction η. Once one knows that the Mayer expansion converges in some disk the implicit function theorem implies that the virial expansion converges for ρ small. Thus Lebowitz and Penrose [5, footnote 18] proved that the virial expansion converges for $\eta \leq 2^{-d}(1+e)^{-1} = 0.034$ in three dimensions. There is numerical evidence for a fluid-to-solid transition at a fluid density $\eta_0 \approx 0.49$, but this is ten times larger! Clever improvements in the Lebowitz-Penrose estimate have been made, but they are not going to help because Groeneveld [4] proved that the Mayer expansion is alternating and this implies that the singularity that limits the convergence of the Mayer expansion is on the negative z axis. Therefore it is not the physical phase transition mentioned above. This negative-axis singularity has acquired the disparaging epithet *unphysical*. Perhaps it is not present in the virial expansion? In other words, the singularity is common to the expansions of ρ and P and might be cancelled when P is expressed as a function of ρ. Numerical evidence based on computing many virial coefficients supports this conjecture. See, for example, Rohrmann et. al. [8]. The first step towards a rigorous result is to find any proof of convergence of the virial expansion that bypasses the Mayer expansion. Thinking of (12) as "sum over connected graphs is less than sum over minimally connected graphs" makes me hope for an analogue where the sum over irreducible graphs is bounded by a sum over not too many minimally irreducible graphs.

Meanwhile the unphysical singularity has had some rehabilitation. It is in fact physical, but for a different system. When Penrose proved his inequality, he actually proved a formula that says that the sum over all connected graphs on n vertices is equal to the sum over just tree graphs on n vertices with more complicated weights. By a different tree formula the complicated weights are revealed in [2] to arise from a natural hard sphere construction in two higher dimensions: the Mayer expansion is term by term equal to the sum over all configurations of a single connected "molecule" formed by glueing hard spheres in $\mathbb{R}^{d+2}$ together in tree topologies, keeping the hard sphere condition. There are pictures of these objects and new developments at

http://www.math.brown.edu/~rkenyon/gallery/gallery.html.

The singularity formerly known as unphysical is at the transition where the molecule is about to become infinitely large. The sign of z is reversed in this correspondence so that the negative z becomes positive.

For the hard sphere system, the age of miracles is not past. Like the Ising model it is one of those canonical models that is on the way to many other problems. It is solvable in one dimension. In two dimensions, the pressure for hard hexagons on the triangular lattice has been computed exactly by R. Baxter [1]. It has a phase transition from a low density phase that probably resembles hard disks. The lattice asserts itself in the high density side of the phase transition and there are three pure phases that correspond to which sublattice is mainly occupied. Does this solution tells us where the virial expansion for this model converges?

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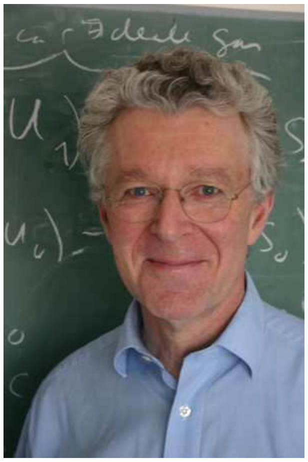
The ESI open to Business on a new Path

by Jakob Yngvason (Vienna, Austria)



Jakob Yngvason is professor of mathematical physics at the University of Vienna and Scientific Director of the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna. After graduating from high school in Reykjavik, he studied physics at Göttingen University, obtaining his Diploma in physics in 1969 and Dr.re.nat. in 1973 (thesis advisor Hans-Jürgen Borchers). Since 1996 he has been professor at the University of Vienna. From 2000-2005 Jakob Yngvason was Vice-President of the International Association of Mathematical Physics and has been Editor-in-Chief of Reviews in Mathematical Physics from 2006-2010. He has made important contributions to local quantum field theory, thermodynamics, and the quantum theory of many-body systems, in particular cold atomic gases and Bose-Einstein condensation. He is co-author, together with Elliott H. Lieb, Jan Philip Solovej and Robert Seiringer, of an important monograph on Bose gases. For his work on Thermodynamics he received, together with Lieb, the Levi Conant Prize of the American Mathematical Society in 2002. In 2004 he received the Erwin Schrödinger Prize of the Austrian Academy of Sciences.

As many readers of the Bulletin will know, the Erwin Schrödinger International Institute for Mathematical Physics in Vienna (ESI) faced a serious threat to its existence last year when the Austrian Government announced in November a decision to terminate funding of more than 70 scientific institutions, including the ESI, as of January 1st 2011. Overwhelming support for ESI's case by the international mathematics and physics community, including a large number of IAMP members, was decisive in averting the worst case scenario for the institute, and at present the ESI seems to be sailing into calmer waters. An agreement has recently been signed between the University of Vienna and the Austrian Ministry of Science to the effect that the ESI will become a research center ("Forschungsplattform") of the University beginning June 1st 2011, with the same name and at same location as before. On the positive side the Ministry has promised to fund the new Forschungsplattform by an earmarked contribution to the university budget until 2014 and possibly 2015. Also, the general goal of the Institute to promote research in mathematics and physics at the highest level with emphasis on fruitful interactions between these disciplines remains unchanged. On the negative side is a substantial cut in the previous funding on part of the Ministry, and the long term consequences of the termination of ESI as an independent institute and its integration into the University of Vienna are not foreseeable at this moment.

To put these developments into perspective it is appropriate to look briefly back at the founding history of the ESI and the development of the institute since its official initiation in 1993. The ESI was created in a political window of opportunity shortly after the fall of the Iron Curtain. An important aspect was the idea to establish an institute in Vienna where scientists from the former Soviet Union and other East European countries could meet with colleagues from the West and thus help to reduce the brain drain from the East. The activities of the institute, however, quickly surpassed this limited goal and ESI established itself as the highly regarded international research center it is today with more than 700 visitors yearly. From the outset the ESI had no permanent scientific staff and focused on thematic research programs with international organizers and participants. The legal structure was that of a society under Austrian law, independent of other academic institutions in Austria, but subsidized by the Ministry of Science on the basis of yearly applications. The activities were gradually broadened with the establishment of a Senior Research Fellows Programme and a Junior Research Fellows Programme. Under the former renowned scientists stay at ESI for several months and give lecture courses for graduate students and postdocs. The Junior Research Fellows Programme supported young predocs and postdocs that participated in the activities of the ESI and interacted with the Senior Fellows and scientists from the local community. The ESI has been evaluated by international panels three times since its founding, the last evaluation being in 2009. All evaluations attested the ESI highest scientific standards and an astonishing cost efficiency.

The transition of the ESI into a research center of the University brings obvious risks and it is particularly deplorable that the highly successful Junior Research Fellows Programme can no longer be upheld due to lack of funds. Also the administrative details of the transition are not at all trivial and have not been completely sorted out yet. The University has, however, been very cooperative and there are reasons to expect that the visitors of ESI will not notice much change. The Society Erwin Schrödinger International Institute for Mathematical Physics will continue to exist and promote the Institute. For the time being we can only hope that the ESI will fare well on its new path. The most important message to IAMP members, however, is that ESI is open to business and in particular welcomes proposals for future programs, see http://www.esi.ac.at/call/call.html.

Report on the Fourth School and Workshop on "Mathematical Methods in Quantum Mechanics" Bressanone, 14-19 February, 2011

The School and Workshop on "Mathematical Methods in Quantum Mechanics" usually takes place every two years in Bressanone, during the month of February. The Baroque town of Bressanone (Brixen in German language) is a lovely town located in the heart of the Italian alps, it is the oldest town in Tyrol and has been a bishop's seat since 901.

View of the inner Court of Bressanone Bishops' Palace

The first three sessions took place in the year 2005, 2007 and 2009, and the fourth session took place from February 14 to February 19, 2011.

The aim of the meeting is to present the state of the art in some challenging open problems in Quantum Mechanics from the point of view of Mathematical Physics. It is mainly addressed to young people interested in working on the subject.

Among the topics covered in the fourth session: adiabatic and semiclassical methods, classical behaviour in quantum systems, effective equations for infinite particle systems, molecular quantum dynamics, nonlinear Schrödinger equations, quantum chaos, quantum chemistry, quantum systems with magnetic fields, quantum transport theory, scattering and spectral analysis for Schrödinger operators.

Three courses were given in a series of lectures scheduled in the morning of each day:

- Luigi Ambrosio (Scuola Normale Superiore - Pisa): Stability of flows and convergence of Wigner transforms
- Gian Michele Graf (ETH Zürich): Transport in quantum pumps and in other open systems
- Christian Lubich (Tübingen Universität): Variational approximations in molecular quantum dynamics

Moreover five invited speakers gave plenary talks:

- Boris Dubrovin (SISSA - Trieste): On critical behaviour in Hamiltonian PDEs
- Paolo Facchi (Università di Bari): Quantum Zeno dynamics: theoretical aspects and applications
- Alessandro Giuliani (Università di Roma 3): Universal conductivity in graphene with short-range interactions
- Mathieu Lewin (Université de Cergy-Pontoise): Geometric methods for nonlinear quantum many-body systems
- Thierry Paul (École Polytechnique - Paris): Rough classical limits

Finally, two parallel sessions were organized for 40 short contributed talks given by participants.

The pdf-presentations of the courses and of the plenary talks are available at the web page of the meeting http://www.mmqm.unimore.it.

During the first day of the conference we commemorated the colleague and friend Naoufel Ben Abdallah, who recently passed away. He participated and gave a course in the previous session.

The total number of participants to the fourth session was 104, coming from several countries: Austria (5), Argentina (2), Brazil (1), Czech Rep. (7), France (10), Finland (2), Germany (26), Italy (40), Ireland (1), Japan (5), Sweden (1), Switzerland (2) and United Kingdom (1).

The financial support obtained from different sponsors by the organizing committee was sufficient for the ordinary expenses of the meeting and also to cover the lodging expenses for 30 young researchers. Among the sponsors of the meeting we would like to mention the contribution of the International Association of Mathematical Physics (IAMP).

Andrea Sacchetti and Alessandro Teta
(for the organizing committee)

Call for prize nominations for prizes to be awarded at ICMP 2012

The Henri Poincaré Prize

The Henri Poincaré Prize, sponsored by the Daniel Iagolnitzer Foundation, was created in 1997 to recognize outstanding contributions in mathematical physics, and contributions which lay the groundwork for novel developments in this broad field. The Prize is also created to recognize and support young people of exceptional promise who have already made outstanding contributions to the field of mathematical physics. The prize is awarded every three years at the International Congress of Mathematical Physics (ICMP), and in each case is awarded usually to three individuals. A list of previous winners can be found here: http://www.iamp.org/page.php?page=page_prize_poincare. Nominations for the 2012 Prize, which is to be awarded at the ICMP in Aalborg, can be made to the President (president@iamp.org) or Secretary (secretary@iamp.org). There is no prescribed format for a nomination but the following information will help the Prize Committee with its task:

- listing or description of the scientific work in support of the nomination
- a recent c.v. of the nominee
- a proposed citation should the nominee be selected for an award

Please use email for your nomination; you will receive an acknowledgment that your nomination has been received.

To ensure full consideration please submit your nominations by November 30, 2011.

The IAMP early career award

The IAMP Early Career Award was instituted in 2008 and was awarded the first time at the ICMP in Prague in 2009. It will be awarded next time at the ICMP in Aalborg in August 2012. The prize rules:

- The prize will be awarded in recognition of a single achievement in mathematical physics. The total prize value amounts to 3000 Euro and is reserved for scientists whose age in years since birth on July 31 of the year of the Congress is less than 35
- A nomination for the prize should include the name of the candidate accompanied by a brief characterization of the work singled out for the prize. Members of the IAMP should send their nomination or nominations to the President (president@iamp.org) and to the Secretary (secretary@iamp.org) by e-mail, and a fax or a hard copy should follow the e-mail

- The recipient of the prize will be selected by a committee of 5 members determined by votes within the Executive Committee of the IAMP, which shall decide the membership of the committee within one month of the expiration of the deadline for nominations
- The Committee may consider candidates that have not been nominated by the members. The names of committee members will remain confidential within the EC until the prize ceremony. The IAMP will make every reasonable effort to make attendance at the ceremony possible for the selected candidate

Nominations should be made not later than on February 6, 2012.

The IUPAP Young Scientist Prize in Mathematical Physics

The IUPAP prize (http://www.iupap.org) is awarded triennially to at the most three young scientists satisfying the following criteria:

- The recipients of the awards in a given year should have a maximum of 8 years of research experience (excluding career interruptions) following their PhD on January 1 of that year
- The recipients should have performed original work of outstanding scientific quality in mathematical physics
- Preference might be given to young mathematical physicists from developing countries

The awards will be given for the second time at the ICMP 2012 in Aalborg; the deadline for nominations is August 2, 2011. Please submit your nomination to Ana Bela Cruzeiro (abcruz@math.ist.utl.pt) and to Herbert Spohn (spohn@ma.tum.de) as officers of the IUPAP C18 commission for mathematical physics.

Pavel Exner
Jan Philip Solovej
Herbert Spohn
Ana Bela Cruzeiro

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. Denis Borisov, Faculty of Physics and Mathematics, Bashkir State Pedagogical University, Ufa, Russia
2. Vojkan Jaksic, Department of Mathematics, McGill University Montreal, Canada
3. Gregory Falkovich, Department of Physics, Weizman Institute of Science, Rehovot, Israel
4. Michael Damron, Department of Mathematics, Princeton University, Princeton, USA
5. Jean-Bernard Bru, Departamento de Matemáticas, Universidad del Pais Vasco and Ikerbasque (Basque Foundation for Science), Bilbao, Spain
6. Walter de Siqueira Pedra, Institute of Mathematics, University of Mainz, Mainz, Germany
7. Annemarie Luger, Department of Mathematics, Stockholm University, Stockholm, Sweden

Open positions

- Postdoctoral position in Random Matrix Theory and Related Fields at the Department of Applied Mathematics, Holon Institute of Technology, Israel.
  http://www.hit.ac.il/ac/files/Eugene.Kanzieper/research/research.jobs.htm
  Deadline: Jun 30, 2011

Recent conference announcements

With support from IAMP:

- Jun 30 – Jul 3, 2011, Signatures of Quantumness in Complex Systems, the University of Nottingham, School of Mathematical Sciences,Nottingham, UK
  http://www.maths.nottingham.ac.uk/personal/ga/SQCS/

Other conferences:

- Aug 16–24, 2011 Summer school on current topics in Mathematical Physics, Erwin Schrödinger Institute, Vienna
  http://www.maphy.uni-tuebingen.de/~chha/Home.html

- May 25–27, 2011 Operator theory and boundary value problems, Université Paris-sud, Orsay, France
  http://orsay2011.info/

Other announcement from the executive committee:

The Henri Poincaré Prizes sponsored by the Daniel Iagolnitzer Foundation and the IAMP early career award will be awarded at the ICMP 12 in Aalborg August 2012. A detailed call for nominations can be found earlier in the bulletin.

Jan Philip Solovej (IAMP Secretary)