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Recent books by IAMP members

With this article, the IAMP News Bulletin launches a new section, where IAMP members get the opportunity to present their recent books by an extended review. The decision whether to publish such a self-promotional review is taken by the editor of the Bulletin in consultation with the editorial board.

A New Book in Harmonic Analysis

by Camil Muscalu and Wilhelm Schlag (Ithaca and Chicago)

Harmonic analysis is an old subject. Its origins lie in the ideas of Fourier in the early 19th century (that were preceded by work of Euler, Daniel Bernoulli, and others) which were revolutionary at the time time and could not be fully understood by Fourier and his contemporaries. However, it was clear even then that the idea of representing any function as a superposition of elementary harmonics (sine and cosine) along an arithmetic sequence of frequencies had important applications to the partial differential equations of physics that were being investigated at the time, such as the heat and wave equations. In fact, it was precisely the desire to solve these equations that lead to this bold claim in the first place.

Research into the precise mathematical meaning of such Fourier series consumed the efforts of many mathematicians during the entire 19th as well as much of the 20th centuries. Many ideas that took their beginnings and motivations as part of the Fourier series research became disciplines in their own right. Set theory (Cantor) and measure theory (Lebesgue) are clear examples, but others such as functional analysis (Hilbert and Banach spaces), spectral theory of operators, the theory of compact and locally compact groups and their representations, all exhibit clear and immediate connections with Fourier series and integrals. Furthermore, soon after Fourier proposed representing every function on a compact interval as a trigonometric series, this idea was generalized by Sturm and Liouville to expansions with respect to the eigenfunctions of very general second oder differential operators subject to natural boundary conditions – a groundbreaking result in its own right.

Not surprisingly, harmonic analysis is therefore a vast discipline of mathematics that continues to be a vibrant research area to this day. In addition, over the past 60 years Euclidean harmonic analysis as represented by the schools associated with A. Calderón and A. Zygmund at the University of Chicago, as well as C. Fefferman and E. Stein at Princeton University, has been inextricably linked with partial differential equations (PDE). While applications to the theory of elliptic PDEs and pseudo-differential operators were one of the driving forces in the development of the Calderón-Zygmund school from the very beginning, the past 25 years have also seen an enormous influx of harmonic analysis techniques to the theory of nonlinear dispersive equations, such as the Schrödinger and

wave equations. These developments continue to this day.

There exist excellent, and by now classic, monographs and textbooks in harmonic analysis such as those by E. Stein [6], [7], and E. Stein, G. Weiss [9]; amongst the older literature there is the timeless encyclopedic work on Fourier series by A. Zygmund [13], and the more accessible introduction to Fourier analysis by I. Katznelson [4]. Various excellent more specialized texts are also available such as G. Folland [2], which focuses on phase space analysis and the Weyl calculus, as well as C. Sogge [5] which covers oscillatory integrals. Wolff's notes [12] can serve as an introduction to the circle of ideas associated with the Kakeya problem. This leads into a more geometric, as well as combinatorial, aspect of harmonic analysis which is still rather poorly understood.

Our book does not intend to compete with any of these well-known texts. From the very beginning it was designed as a teaching tool, which can mean both in the traditional classroom setting, as well as in the setting of independent study by an advanced undergraduate or beginning graduate student. In addition, the authors hope that it will also be useful for any scientist who wishes to acquire working knowledge in select topics in (mostly Euclidean) Fourier analysis. Finally, since Volume 1 begins with some rather introductory material in analysis, it can be used selectively also as an undergraduate textbook.

The two volumes of our book are quite different in both scope and character, although they should be perceived as forming a natural unit. The first volume attempts to introduce the reader to a broad array of results and techniques, starting from the beginnings of the field (Fourier series), and then developing the theory along what the authors hope are natural avenues motivated by certain basic questions. The selection of the material chosen for the first volume is of course a reflection of the authors' tastes, but it also follows a specific purpose: to introduce the reader to sufficiently many topics in classical Fourier analysis, and that in enough detail, so as to allow them to continue guided study of more advanced material — possibly leading to original research in analysis, both pure and applied (including mathematical physics).

All of the material in the first volume should be considered basic. It can be found in many texts, but to the best of our knowledge not in a single place. The authors feel that Volume 1 represents the course that they should have taken in graduate school, at least based on hindsight. To what extent the entirety of Volume 1 makes for a reasonable class is up to the individual teacher to decide. It is more likely that selections need to made, and it is of course also to be expected that lecturers will wish to supplement the material with certain topics of their own choosing which are not covered here. But the authors feel strongly, in particular since this has been tested on individual students, that Volume 1 can be covered by a beginning graduate student over the course of a year in independent, but guided, study. This would then culminate in some form of "qualifying" or "topic" exam, after which the student would be expected to begin independent research.

Particular emphasis has been placed on the inclusion of exercises and problems. The former are dispersed throughout the main body of the text, and are for the most part an integral part of the theory. As a rule they are less difficult than the problems collected at the end of the chapters. Those problems serve to develop the theory further, and to

give the reader the opportunity to try his or her hand at the occasional "hard" problem. An old and commonplace principle, to which the authors adhere, is that any piece of mathematics can only be learned by active work, and this is reflected in our book. In addition, the authors have striven to emphasize intuition and ideas over both generality and technique, without, however, sacrificing rigor, elegance, or for that matter, relevance.

While Volume 1 only presents mathematics that had been known by the mid to late 1980s, Volume 2 picks up from there and focuses on some aspects of harmonic analysis that were developed more recently. In contrast to the first volume, the second one is much more selective and many topics of recent and current interest are not included. Examples of the many omitted areas that come to mind are oscillatory integrals related to the Kakeya, restriction, and Bochner-Riesz conjectures, multilinear Strichartz estimates, geometric measure theory and its relations to combinatorics and number theory. In addition, Volume 2 is not an encyclopedic work on multilinear harmonic analysis. Throughout both volumes the authors have striven to emphasize intuition and ideas, and often figures are used as important part of explanations and proofs. This is particularly the case for the second volume, which is more demanding in terms of technique with often longer and more complicated proofs than those that can be found in the first volume.

In terms of prerequisites, a good grounding in basic analysis is of essence; beginning with multi-variable calculus (including how to write a hyper-surface as a graph, and the notion of Gaussian curvature on the surface), then a good grasp of measure theory and integration, some functional analysis in the form of basic Hilbert spaces (orthogonal projections, bases, completeness) as well as Banach spaces (weak and strong convergence, bases, Hahn-Banach, uniform boundedness), and finally some basic complex analysis (holomorphic functions and conformal maps). In the first volume probability theory also makes an appearance, and it might be helpful if the reader has had some exposure to the notion of independence, expected value, variance and distribution functions. But for the most part, we require nothing that might be considered "sophisticated". The second volume requires very little in terms of preparation, other than maturity. The authors do not recommend, however, to attempt the second volume before the first one.

As is customary in analysis, interpolation is frequently used. To be more specific, we rely on both the Riesz-Thorin and Marcinkiewicz interpolation theorems. We state these facts at the end of the first chapter, but omit the proofs as they can be readily found in the texts of Stein and Weiss [9], as well as Katznelson [4]. Finally, in the chapter on the restriction theorem we also make use of Stein's generalization of the Riesz-Thorin theorem to analytic families of operators. The second volume uses multilinear interpolation, and the needed facts are collected in an appendix.

As we have already pointed out, many important topics are omitted from our two volumes. Apart from classical topics such as inner and outer functions which we could not include in Volume 1 due to size considerations, an entire aspect of "modern" harmonic analysis that is not covered here is the vast area dealing with "oscillatory integrals". This field touches upon several other areas such as geometric measure theory, combinatorial geometry and number theory, and is relevant to nonlinear dispersive PDEs in several ways such as through bilinear restriction estimates (see for example Wolff's paper [11],

as well as his notes [12]). This material would naturally comprise a third volume, which would need to present the research that was done after Sogge [5] and Stein [8]. However, for the moment such a volume is not in the works.

We now discuss the second volume in more detail, especially in terms of the historical developments of the subjects it presents. As already mentioned above, it is more specialized in the sense that it is entirely devoted to multi-linear aspects of singular integrals and pseudo-differential operators. However, at the same time it is also broad as it aims to cover a wide range of topics within that area. By design, each of these topics is presented within the framework of a few overarching principles. Amongst these the most fundamental notion is that of a *paraproduct*, and we devote the first three chapters of Volume 2 to the introduction, motivation, and development of this basic idea. The immediate goal that these three chapters aim for is a systematic and unifying treatment of fractional Leibnitz rules. In the later chapters which analyze more difficult operators such as the Carleson maximal function and Calderón commutators, it will become necessary to understand the combinatorics involved in handling many paraproducts simultaneously.

While paraproducts already make an appearance in Volume 1 in the context of Haar functions and their use in the proof of the $T(1)$ theorem, it is only in Volume 2 that we delve more deeply into the structure of these objects.

The core of Volume 2 consists of three main strands which are very much interwoven:

a) Calderón commutators and the Cauchy integral on Lipschitz curves

b) The bilinear Hilbert transform

c) Carleson's theorem of the almost everywhere convergence of L^2 Fourier series

While the relation between topics a) and b) had been observed many years ago by Calderón, who also introduced them into harmonic analysis, the close relation between b) and c) is a more recent discovery.

Let us now give a brief synopsis of the history of each of these topics. In his thesis of 1915, N. N. Lusin conjectured that the Fourier series of any L^2-function converges almost everywhere. The question of whether or not this is true proved to be very difficult to answer, and out of the reach of "classical" methods. In 1922 A. N. Kolmogoroff famously constructed an example of an L^1 function for which the associated Fourier series diverges almost everywhere. In an attempt to shed some light on the L^2 case, Paley and Zygmund studied random Fourier series and established the theorem that an L^2 series with random coefficients converges almost surely almost everywhere; see Chapter 6 of Volume 1 for both Kolmogoroff's example and random Fourier series.

It was not until 1966 that L. Carleson established Lusin's conjecture as correct. The L^p-version of Carleson's theorem for $1 < p < \infty$ was obtained later on by Hunt. Carleson's theorem is based on a deep phase-space analysis of L^2 functions.

If convergence in L^2 of Fourier series is equivalent to the L^2 boundedness of each partial sum operator $S_N f(x)$ independently of N (which is an easy consequence of the Plancherel theorem), it is natural to expect that proving almost everywhere convergence requires handling infinitely many such (localized) partial Fourier sums simultaneously.

This is a very difficult task since any collection of partial sums may overlap with each other both in space and in frequency. To prove his theorem, Carleson invented a very delicate combinatorial and analytical process of organizing these partial sums, based on the geometric intuition coming from the phase space picture and the Heisenberg principle, and which rendered these carefully selected partial sums almost orthogonal.

A few years after Carleson's breakthrough, C. Fefferman gave a new proof of the Carleson-Hunt theorem by building upon as well as simplifying some of these ideas. In addition, he also introduced the modern language of tiles and trees that has been used in the field ever since. A proof of the Carleson-Hunt theorem is given in Chapter 7 of Volume 2.

Around the same time, Calderón introduced commutators and the Cauchy integral on Lipschitz curves as part of his program in PDE's. These interesting and natural singular integrals are no longer of a convolution type and cannot be treated by means of the earlier Calderón-Zygmund theory. By using a combination of methods from complex and harmonic analysis, Calderón proved the L^p boundedness of his first commutator in 1965, but his method could not handle the second commutator let alone the higher ones. In spite of efforts by numerous harmonic analysts, these problems resisted solution until 1975 when Coifman and Meyer proved the desired bounds for the second commutator and shortly afterwards for all Calderón commutators. Their proof builds on some of Calderón's ideas, but it was based only on harmonic analysis techniques. These authors were the first to realize in a profound way that the commutators are in fact multi-linear operators and their method of proof used this observation crucially. Around the same time they proved what is now called the Coifman-Meyer theorem on paraproducts. It is also interesting to note that all these multi-linear operators came out of studies of linear PDE problems. A few years later, however, Bony realized that paraproducts play an important role in non-linear PDEs — which they do to this day.

After all this progress, the Cauchy integral on Lipschitz curves (which initially appeared in complex analysis) was the last operator that remained to be understood. The Cauchy integral possesses the remarkable feature of being naturally decomposed as an infinite sum of all Calderón commutators with the simplest (linear) term equaling the classical Hilbert transform. In order to prove estimates for it one therefore needed to prove sufficiently good bounds for the operator norms of all the commutators, so as to be able to sum up all of their contributions. The initial proof by Coifman-Meyer was very intricate and it was not at all clear what type of bounds it yielded for the commutators. Calderón was the first to observe that there is a proof which gives exponential bounds, thus proving L^p estimates for the Cauchy integral under the assumption that the Lipschitz constant is small. It had also been observed that polynomial bounds for the commutators would allow for a complete understanding of the Cauchy integral on Lipschitz curves and many of its natural extensions. The final breakthrough was achieved in 1982 by McIntosh, Coifman, and Meyer who established the desired polynomial bounds for Calderón commutators. Chapter 4 of Volume 2 is devoted to the proofs of all these results. The proofs we give there differ from the original literature.

Prior to the resolution of his conjectures on the commutators, Calderón proposed the

study of the bilinear Hilbert transform. He viewed this as a step towards the commutators and eventually of the Cauchy integral on Lipschitz curves. It had been observed that the first commutator was equal to an average of such bilinear operators. As we now know, the whole Calderón program could have been be completed in a different manner, but the question of whether the bilinear Hilbert transform satisfies any L^p estimates remained open. It was finally solved by Lacey and Thiele in two papers from 1997 and 1999. As it turned out, the analysis of the bilinear Hilbert transform is very closely related to the analysis of the Carleson maximal operator, which appeared implicitly in the proof of almost everywhere convergence of Fourier series.

A brief explanation for this is as follows. When viewed as a bilinear multiplier, the symbol of the bilinear Hilbert transform is singular along a one-dimensional line. This line is responsible for the one-dimensional modulation symmetry of the bilinear Hilbert transform, which is identical to the modulation symmetry of the Carleson operator. In particular, both of these objects have precisely the same symmetries: translation, dilation and modulation invariances. By comparison, paraproducts have classical Marcinkiewicz-Mikhlin-Hörmander symbols and these are smooth away from the origin, in other words they have a zero-dimensional singularity. They are only translation and dilation invariant, which are the usual symmetries of the classical Calderón-Zygmund convolution operators. A proof of the boundedness of the bilinear Hilbert transform appears in Chapter 6 of Volume 2.

Let us conclude with a few words about Chapter 5, which describes the more recent theory of iterated Fourier series and integrals. This chapter is almost entirely devoted to motivation and is somewhat speculative in character. To be more specific, we describe what appears to be a very natural "physical" problem where both the Carleson maximal operator and the bilinear Hilbert transform appear simultaneously. In fact, they are the simplest operators in an infinite series which determines the solutions of a certain ODE. This problem goes beyond *multilinear harmonic analysis* and may be said to belong to *nonlinear harmonic analysis*, since the study of its multilinear building-blocks is not sufficient for its complete understanding. It is towards such a theory of nonlinear harmonic analysis that Chapter 5 aspires.

How to use Volume 1: The chapters are ordered linearly, and should ideally be worked through in that order. There is a certain amount of dependence, which can be offset with prior experience. To be more specific, a reader or class familiar with the Fourier transform in $\mathbb{R}^d$ can start with Chapter 7, then move on the Chapter 8, and subsequently choose any of the remaining four chapters according to taste and time constraints. Chapters 7 through 10 constitute the back-bone of "real-variable" harmonic analysis. Of those, Chapter 10 can be regarded as more of an outlier as it is mostly not included in such a setting. However, the authors feel that it is of importance and students should be exposed to it. As an application of the Logvinenko-Sereda theorem of Section 10.3, we prove the local solvability of constant coefficient PDEs (the Malgrange-Ehrenpreis theorem) in Section 10.4.

Another such outlier is Chapter 12, which attempts to introduce the reader to an area that in an of itself is the subject of many books, see for example Taylor's books on pseudo-

differential operators [10], as well as Hörmander's treatise [3]. The authors decided to included a very brief account of this story, as it is an essential part of harmonic analysis and also since it originates in Calderón's work as part of his investigations of singular integrals and the Cauchy problem for elliptic operators. In principle, this chapter can be read separately by a mature reader who is familiar with Cotlar's lemma from Chapter 9. In the writing of this chapter a difficult decision had to be made, namely which of the two main incarnations of pseudo-differential operators to use, Kohn-Nirenberg or Weyl. While the former is somewhat simpler technically, and therefore often used in elliptic PDEs, the older Weyl quantization is very natural due to its symmetry, and mostly used in the so-called semiclassical calculus. We therefore chose to follow the latter route, with Kohn-Nirenberg pseudo-differential operators making only a very brief appearance in this text.

Chapter 5 introduces the reader to probability theory, which is also often omitted in a more traditional harmonic analysis class. However, the authors feel that the ideas developed in that chapter (which are very elementary for the most part) are an essential part of modern analysis, and mathematics in general. They appear in many different settings and should be in the toolbox of any working analyst, both pure and applied. Chapter 6 contains several examples of how probabilistic thinking and results appear in harmonic analysis. Section 6.3 on Sidon sets can be omitted on first reading, as it is indeed somewhat specialized (in particular, the theorem of Rider's which we prove there).

The first four chapters of Volume 1 are intended for a reader who has had no or very little prior exposure to Fourier series and integrals, harmonic functions, and their conjugates. A basic introductory advanced undergraduate or beginning graduate course would cover the first three chapters without Section 2.5, and then move on the first section of the fourth chapter (the material on locally compact Abelian groups can be omitted entirely — it is used only in the non-Euclidean setting in the proof of Rider's theorem in Section 6.3 — and stationary phase is used only in the final two chapters of Volume 1). After that, an instructor can then choose to select any topics from Chapters 5–8 as desired, the most traditional choice being the Calderón-Zygmund, Mikhlin, and Littlewood-Paley theorems. The latter, at least as presented here, does require a minimum of probability, namely in the form of Khinchine's inequality.

How to use Volume 2: The are several options. For instance, after completing Chapters 7 and 8 of Volume 1 on the classical theory of singular integrals, it is natural to move on the more advanced topics presented in Volume 2 such as the theory of Calderón commutators and the Cauchy integral on Lipschitz curves. We would like to emphasize that this is indeed possible since Chapter 4, where these topics are covered, only assumes familiarity with very basic harmonic analysis such as maximal functions, Calderón-Zygmund operators, and Littlewood-Paley square functions. It does not rely on any other material from Volume 1 such as the $T(1)$ theorem, BMO, or Carleson measures.

On the other hand, for an audience which is more inclined towards learning techniques useful for nonlinear PDEs, it might be advisable to focus on paraproducts and Leibnitz rules. In that case, Chapters 1, 2, 3 and 8 would be the natural order to proceed.

A more mature reader wishing to study the almost everywhere convergence of Fourier

series can in principle start with Chapter 7. However, that chapter does rely on some technical results from Chapter 6, where the bilinear Hilbert transform is presented. We therefore recommend that Chapter 7 should be attempted only after mastering Chapter 6. That chapter, on the other hand, can be read independently by anyone familiar with paraproducts and the John-Nirenberg inequality (which are covered in Chapter 2).

Finally, we would like to stress that Volume 2 was designed with the specific purpose of fitting seemingly disparate objects into a unifying frame work. The clarity and transparency that we hope to have achieved by doing so will only be appreciated by the patient reader willing to take the journey from beginning to end.

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Classical and Multilinear Harmonic Analysis,Cambridge University Press

Three stories of high oscillation

by Arieh Iserles (Cambridge)

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Think globally, act globally!

Numerical analysis – indeed, mathematical analysis – is in large measure a story of the local. Sufficiently smooth functions are approximated very well in a small neighbourhood by polynomials, indeed even by linear functions. A case in point is the granddaddy of all discretisation methods for differential equations, the *Euler method:* Given the ordinary differential system $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ and having already found $\boldsymbol{y}_n \approx \boldsymbol{y}(t_n)$, we approximate $\boldsymbol{y}(t_{n+1})$, where $t_{n+1} = y_n + h$, by $\boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$. In other words, we progress from t_n to t_{n+1} assuming that locally the solution is linear, uniquely determined from its value and its slope at t_n. As long as $\boldsymbol{f}$ is twice smoothly differentiable, the error is $\mathcal{O}(h^2)$ and it can be made sufficiently small by choosing small step-size $h > 0$. Of course, in place of linear functions, we can use polynomials, say, or rational functions of increasing degree of sophistication and intricacy but the same principle applies: we zoom on a sufficiently small neighbourhood. *Reductio ad absurdum* of numerical analysis would leave us with a single phrase, "the Taylor theorem".

As long as functions are 'nice', the Taylor theorem is a reasonable foundation stone to numerical algorithms. It is difficult to imagine the modern world without scientific computing and its many achievements: acting locally to think globally is the right strategy most of the time. Yet, important exceptions abound and they come in two flavours. Some functions are not sufficiently smooth to be approximated well by, say, polynomials, calling for more subtle forms of discretisation. Other functions might be misleadingly 'nice', yet their derivatives are large, rendering their local approximation with the Taylor theorem very poor indeed. This is in particular the case with *highly oscillating functions,* because, once Taylor series are truncated, the error scales like a (high) derivative but, each time we differentiate, the amplitude is multiplied by the frequency.

An accessible example is provided by the *Airy equation* $y'' + ty = 0$. The solution

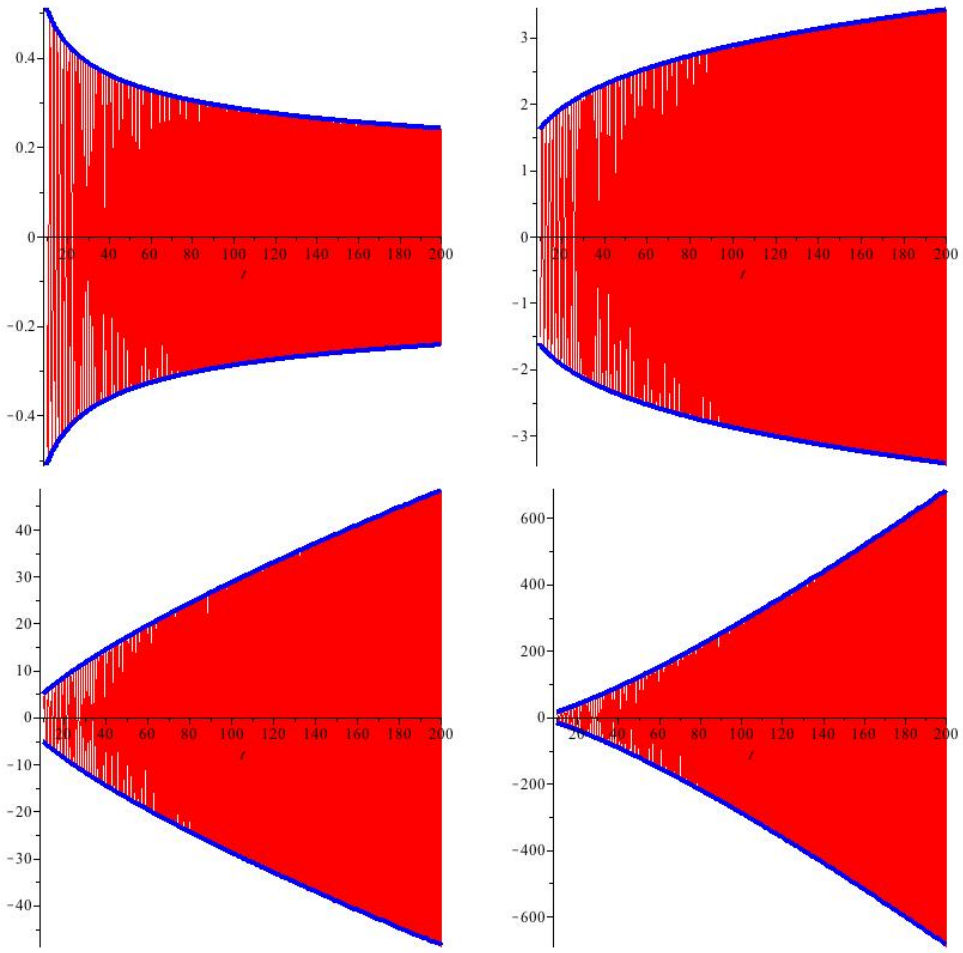

Figure 1.1: The solution of $y'' + ty = 0$, $y(0) = 1$, $y'(0) = 0$ and its first, second and third derivatives. The thick blue line depicts the envelope $\pm 0.91 t^{m/2-1/4}$.

behaves for large t as $y(t) \sim ct^{-1/4} \sin(\frac{2}{3}t^{3/2})$ – it oscillates increasingly fast as t grows [13]. It is easy to confirm that $y^{(m)}(t) \sim ct^{m/2-1/4} \sin(\frac{2}{3}t^{3/2})$, $m \geq 0$, and the amplitude of derivatives increases rapidly for $t \gg 1$. This is demonstrated in Fig. 1.1. Thus, as t grows, $y(t)$ is poorly approximated by polynomials and increasing the polynomial degree actually makes things worse!

High oscillation is present in a very wide range of phenomena and numerical modelling of high-frequency phenomena is vital. Yet, the Taylor theorem alone is inadequate for this purpose, calling for an entirely new breed of discretisation methods. One such type of methods endeavours to bring together numerical analysis and asymptotic expansions, marrying local behaviour with global features of the solution. Three such scenarios are described in this paper: the computation of highly oscillatory integrals, the discretisation of ordinary differential equations with highly oscillatory forcing terms and the computation of the linear Schrödinger equation. In neither case have we tried to achieve the

greatest possible generality or sophistication, opting instead for accessible examples of numerical methods where global behaviour informs computation.

Story 1: Highly oscillatory quadrature

Computing integrals is an art as old as integration itself and the oldest quadrature method is due to Sir Isaac Newton (although, in fairness to Archimedes, he used something very much like quadrature to calculate the area of a disc). This is the lore of undergraduate numerical analysis courses: given a weight function $w(x) \geq 0$, not identically zero, and assuming that everything in sight is smooth,

$$\int_{a_-}^{a_+} f(x)w(x)\,\mathrm{d}x \approx Q_\nu[f] = \sum_{m=1}^{\nu} b_m f(c_m),$$

where the *nodes* $c_1, \ldots, c_\nu$ are distinct numbers in $[a_-, a_+]$, while $b_1, \ldots, b_\nu$ are the quadrature *weights.* Other things being equal, the best nodes are the zeros of the ν-degree orthogonal polynomial with respect to the L_2 inner product generated by the weight function w and the outcome, *Gaussian quadrature,* is exact for all polynomials f of degree $\leq 2\nu - 1$.

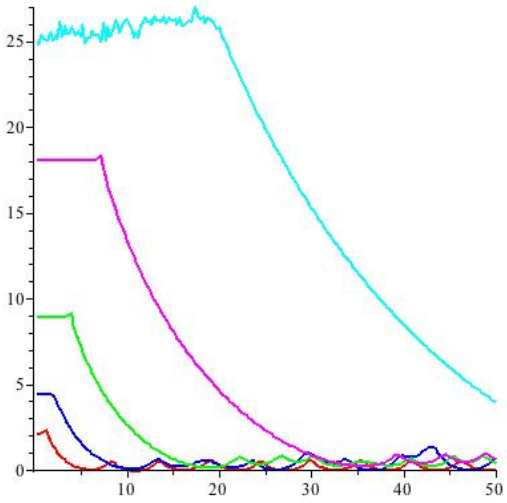

Figure 2.1: The number of significant digits, $-\log_{10}|Q_\nu[f] - I[f]|$, for $f(x) = (2+x)^{-1}$, $\nu = 2$ (red), 4 (blue), 8 (green), 16 (magenta) and 32 (cyan) for increasing ω.

Let

$$I[f](\omega) = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x. \tag{2.1}$$

Once ω becomes large, the integrand in (2.1) oscillates at an increasingly faster pace. In Fig 2.1 we display the magnitude of the error once Gaussian quadrature is applied to

$I[f](\omega)$ for $0 \le \omega \le 50$. The pattern is plain to see: for small ω Gaussian quadrature performs very well and delivers remarkable accuracy: the larger ν, the better. However, once ω grows, roughly when $\omega\nu > 1$, accuracy drops sharply and, after a short while, (2.1) fails to deliver even a single significant digit!

The reason is plain to see: the integrand in (2.1) is very poorly approximated by polynomials of reasonable degree. We are precisely in the regime where the Taylor theorem is of little use! Thus, we turn to asymptotics for help. It is easy to prove that

$$I[f](\omega) \sim -\sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}}[f^{(n)}(1)\mathrm{e}^{\mathrm{i}\omega} - f^{(n)}(-1)\mathrm{e}^{-\mathrm{i}\omega}], \qquad \omega \gg 1,$$

and this indicates the *asymptotic method*

$$A_s[f](\omega) = -\sum_{n=0}^{s} \frac{1}{(-\mathrm{i}\omega)^{n+1}}[f^{(n)}(1)\mathrm{e}^{\mathrm{i}\omega} - f^{(n)}(-1)\mathrm{e}^{-\mathrm{i}\omega}] \tag{2.2}$$

[9]. Note that the error, $\mathcal{O}(\omega^{-s-2})$, improves for increasing ω!

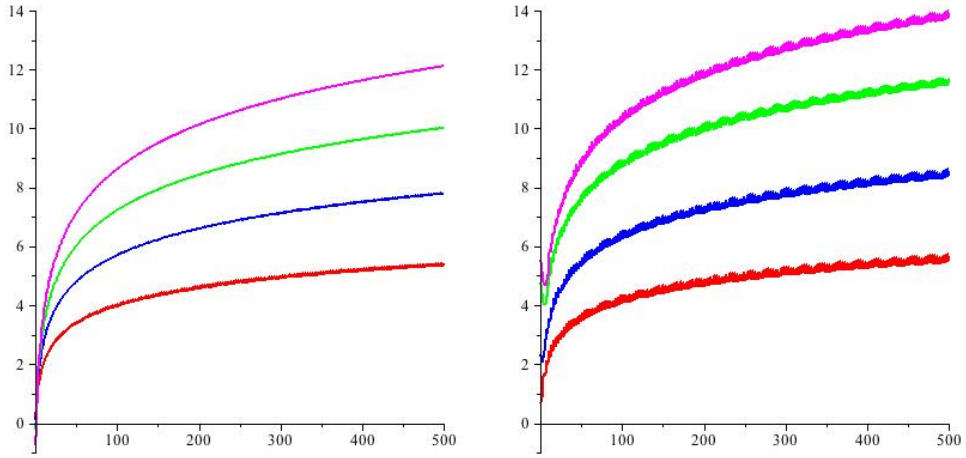

Figure 2.2: The number of significant digits once we use highly oscillatory quadrature. On the left the asymptotic method $A_s[f]$ for $s = 0$ (red), 1 (blue), 2 (green) and 3 (magenta). On the right the Filon-type method $F_{\boldsymbol{c},\boldsymbol{m}}[f]$ with $\boldsymbol{c} = [-1, 1]$, $\boldsymbol{m} = [1, 1]$ (red), $\boldsymbol{c} = [-1, 0, 1]$, $\boldsymbol{m} = [2, 1, 2]$ (blue), $\boldsymbol{c} = [-1, -1/\sqrt{3}, 0, 1/\sqrt{3}, 1]$, $\boldsymbol{m} = [3, 1, 1, 1, 3]$ (green) and $\boldsymbol{c} = [-1, -\sqrt{3/11}, 0, \sqrt{3/11}, 1]$, $\boldsymbol{m} = [4, 1, 1, 1, 4]$.

Fig. 2.2 on the left displays the error (in a format identical to Fig. 2.1) incurred by (2.2) for $s = 0, 1, 2, 3$, except that we have taken a much larger range of ω. The comparison with Gaussian quadrature is striking: for large ω we require *much* less information, yet obtain a surprisingly precise answer. The *quid pro quo* is that, having committed all our assets to recover 'large ω' behaviour, we can hardly complain that the asymptotic method is useless for small $\omega > 0$. In a sense, Gaussian quadrature and the asymptotic method are complementary.

Which brings us to the right-hand side of Fig. 2.2. We see there for large ω behaviour somewhat better than (2.2) but careful examination of the neighbourhood of the origin indicates that the mystery method performs well also for small ω. Since the extra expense of the mystery method, compared with (2.2), is marginal, we indeed enjoy the best of all worlds: an asymptotic–numerical method.

So, what is the mystery method? Given the nodes $c_1 = -1 < c_2 < \cdots < c_{\nu-1} < c_\nu = 1$, each c_k with a *multiplicity* $m_k \geq 1$, there exists a unique *Hermite interpolating polynomial* p of degree $\sum m_k - 1$ such that $p^{(i)}(c_k) = f^{(i)}(c_k)$, $i = 0, \ldots, m_k - 1$, $k = 1, \ldots, \nu$. A *Filon-type method* consists of replacing f by p in the integral,

$$F_{\boldsymbol{c},\boldsymbol{m}}[f](\omega) = I[p](\omega). \tag{2.3}$$

Let $s = \min\{m_1, m_\nu\}$. It is trivial, substituting $F_{\boldsymbol{c},\boldsymbol{m}}[f] - I[f] = I[p - f]$ into (2.2), to verify that the error is $\mathcal{O}(\omega^{-s-2})$ [9]: for large ω the behaviour is determined by asymptotics. On the other hand, once $\omega \to 0$, (2.3) collapses into a quadrature method (using both the values of f and of its derivatives) of the classical kind.

Beyond Filon-type methods It is easy to extend the Filon method to integrals of the kind

$$\int_{a_-}^{a_+} f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x,$$

provided that $g' \neq 0$ in $[a_-, a_+]$. More effort is required to deal with the case of stationary points $\xi \in [a_-, a_+]$ where $g'(\xi) = 0$. In that case we need to interpolate to f and a suitable number of derivatives at stationary points to recover the asymptotic behaviour of the 'plain' Filon-type method [9]. All this extends to a multivariate setting [10].

However, Filon-type methods are just one of the new breed of highly oscillatory quadrature methods, alongside Levin methods [12, 14] and the numerical stationary phase method [8]. The last word is a methodology which combines an insight from both Filon-type and stationary phase approaches in a true tour de force of asymptotics-cum-numerics [7].

Story 2: A highly oscillatory forcing term

A good starting point is the differential equation for a nonlinear, frictionless pendulum which, in a dimensionless form, reads $y'' + \sin y = 0$, $y(0) = y_0$, $y'(0) = y_0'$. This is a Hamiltonian system in a single variable and its dynamics is as simple as they come: the origin is a centre, surrounded in the phase plane by stable periodic orbits. Suppose, however, that we impart rapid oscillations to the base of the pendulum: the outcome is the non-autonomous equation

$$y'' + \sin y = c\cos\omega t, \quad t \geq 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{3.1}$$

where $\omega \gg 1$. What can we expect from the solutions of (3.1)? Fig. 3.1 displays a trajectory corresponding to $y_0 = 2$, $y_0' = 0$ and $c = 1$ in the phase plane for $\omega = 5, 10$ and 20. We can discern a black curve in the background: this is the corresponding trajectory of the nonlinear pendulum $y'' + \sin y = 0$ and it is evident that the trajectory of the

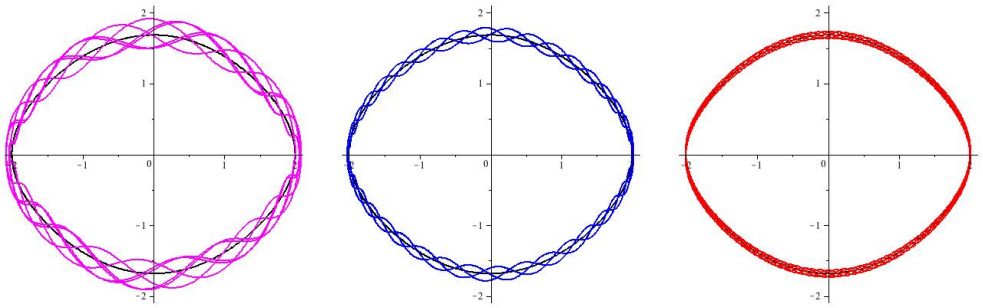

Figure 3.1: A phase-plane trajectory of the solution of $y'' + \sin y = \cos \omega t$, $y(0) = 3$, $y'(0) = 0$, with $\omega = 5, 10, 20$. The black curve is the corresponding periodic trajectory of $y'' + \sin y = 0$.

forced equation winds round and round the periodic trajectory of the unforced equation. Surprisingly, the amplitude of this winding motion *decreases* with ω. (This is the reason why we have used in Fig. 3.1 such modest values of ω: with much greater values the winding motion would have been invisible to the naked eye.)

This effect of growing ω is important: rapid oscillation of the forcing term leads to increasingly smaller perturbation and it stabilises the motion. By the same token, slowing down the oscillation destabilises the motion: try the same computation with $\omega = 2$. There is an important lesson here: high oscillation is good for dynamics while, at the same time, being lethal for naive numerics. True to our paradigm, instead of relying on the (local) Taylor series, we seek an asymptotic expansion.

It is convenient to consider a broader framework: following [3], we examine the equation

$$y'' + f(y) = \sum_{m=-\infty}^{\infty} a_m(t) \mathrm{e}^{\mathrm{i}m\omega t}, \quad t \geq 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{3.2}$$

where f and a_m, $m \in \mathbb{Z}$, are analytic functions and $\sum_m |a_m(t)| < \infty$, $t \geq 0$. Our claim is that we can expand the solution in the form

$$y(t) \sim p_{0,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}, \qquad t \geq 0, \tag{3.3}$$

where the functions $p_{r,m}$, which are independent of ω, can be obtained explicitly in a recursive manner.

In principle, we need to expand everything in (3.2) in asymptotic series. This is fairly easy for $y''(t)$ but much more challenging (and messy) for $f(y(t))$ and, at the conclusion of all this algebra galore, we have an expansion in two scales: *orders of magnitude* ω^{-r} and *oscillators* $\mathrm{e}^{\mathrm{i}m\omega t}$. We commence by separating the expansion into orders of magnitude. For $r = 0$ we have

$$p_{0,0}'' - \sum_{m=-\infty}^{\infty} m^2 p_{2,m} \mathrm{e}^{\mathrm{i}m\omega t} + f(p_{0,0}) = \sum_{m=-\infty}^{\infty} a_m \mathrm{e}^{\mathrm{i}m\omega t}$$

and, separating oscillators, we have

$$\begin{aligned} p''_{0,0} + f(p_{0,0}) = 0, \quad t \geq 0, \qquad p_{0,0}(0) = y_0, \quad p'_{0,0}(0) = y'_0, \\ p_{2,m} = -\frac{a_m}{m^2}, \qquad m \neq 0. \end{aligned} \tag{3.4}$$

Note that we have imposed on $p_{0,0}$ the same initial conditions as on y, this means that we need to impose $\sum_m p_{r,m}(0) = \sum_m p'_{r,m}(0) = 0$ in the sequel.

Next, over to $r = 1$,

$$\sum_{m=-\infty}^{\infty} (2\mathrm{i}mp'_{2,m} - m^2 p_{3,m})\mathrm{e}^{\mathrm{i}m\omega t} = 0$$

and, separating scales,

$$p_{3,m} = \frac{2\mathrm{i}}{m} p'_{2,m} = -\frac{2\mathrm{i}}{m^3} a'_m, \qquad m \neq 0. \tag{3.5}$$

Likewise, $r = 2$ results in

$$\sum_{m=-\infty}^{\infty} (p''_{2,m} - 2\mathrm{i}mp'_{3,m} - m^2 p_{4,m})\mathrm{e}^{\mathrm{i}m\omega t} + f'(p_{0,0}) \sum_{m=-\infty}^{\infty} p_{2,m}\mathrm{e}^{\mathrm{i}m\omega t} = 0,$$

therefore

$$\begin{aligned} p''_{2,0} + f'(p_{0,0})p_{2,0} = 0, \quad t \geq 0, \qquad \sum_{m=-\infty}^{\infty} p_{2,m}(0) = \sum_{m=-\infty}^{\infty} p'_{2,m}(0) = 0, \\ p_{4,m} = \frac{1}{m^2}[p''_{2,m} - 2\mathrm{i}mp'_{3,m} + f'(p_{0,0})p_{2,m}], \qquad m \neq 0. \end{aligned} \tag{3.6}$$

A general rule emerges: for each order of magnitude r we obtain $p_{r,0}$ solving a differential equation and $p_{r+2,m}$, $m \neq 0$, by recursion. These differential equations need be solved numerically, but this is not a problem since they are all non-oscillatory: the solution becomes oscillatory only once we assemble it into an appropriately truncated expansion (3.3),

$$y^{[s]}(t) = p_{0,0}(t) + \sum_{r=2}^{s} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}, \qquad t \geq 0$$

All relevant equations can be derived explicitly [3].

Specialising to the forced nonlinear pendulum (3.1), we use (3.4–6) and, for a good measure, consider also $r = 3, 4$. We have $p_{3,m} \equiv 0$,

$$\begin{aligned} p''_{0,0} + \sin p_{0,0} = 0, \quad t \geq 0, \qquad & p_{0,0}(0) = 2, \quad p'_{0,0}(0) = 0, \\ p''_{2,0} + (\cos p_{0,0})p_{2,0} = 0, \quad t \geq 0, \qquad & p_{2,0}(0) = 1, \quad p'_{2,0}(0) = 0, \\ p''_{4,0} + (\cos p_{0,0})p_{4,0} = \tfrac{1}{4}(1 + 2p_{0,0}^2)\sin p_{0,0}, \qquad & p_{4,0}(0) = \cos 2, \quad p'_{4,0}(0) = 0, \end{aligned}$$

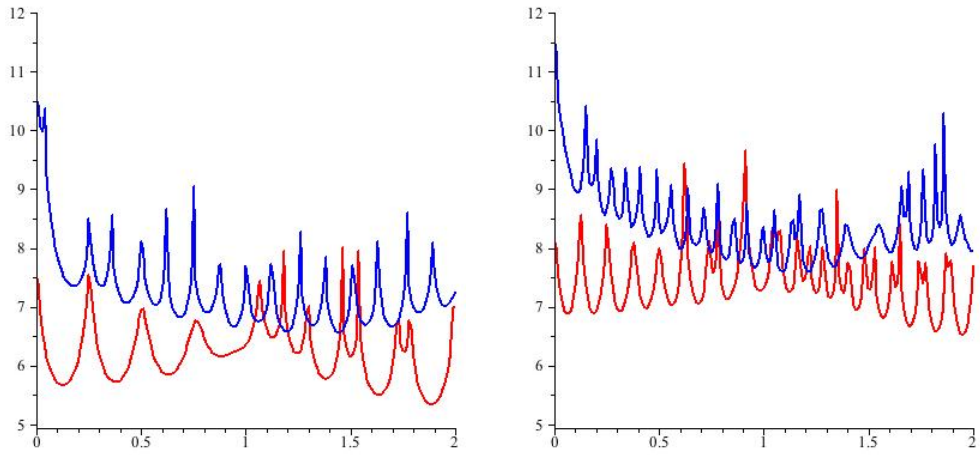

Figure 3.2: The error $-\log_{10}|y^{[s]}(t) - y(t)|$ for the forced nonlinear pendulum (3.1): $\omega = 25$ on the left and $\omega = 50$ on the right, $s = 2$ in red and $s = 4$ in blue.

and

$$y^{[2]}(t) = p_{0,0}(t) + \frac{1}{\omega^2}[p_{2,0}(t) - \cos\omega t],$$
$$y^{[4]}(t) = p_{0,0}(t) + \frac{1}{\omega^2}[p_{2,0}(t) - \cos\omega t] + \frac{1}{\omega^4}[p_{4,0}(t) - \cos\omega t \cos p_{0,0}(t)].$$

Fig. 3.2 displays the number of significant digits recovered by $y^{[2]}$ and $y^{[4]}$, respectively, for $\omega = 25$ and $\omega = 50$. All is in line with the theory and if the accuracy is insufficient for you – well, just take larger ω...

Beyond highly oscillatory forced ODEs The equation (3.2) is but one example of ordinary differential equations with highly oscillatory forcing which can be expanded by this synthesis of asymptotic and numerical reasoning. With greater generality, it is possible to extend this analysis to ODE systems

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \sum_{m=-\infty}^{\infty} \boldsymbol{a}_m(t)\mathrm{e}^{\mathrm{i}m\omega t}, \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

[4, 15] and beyond. Equations of this kind are important in many applications, e.g. in electronics, where oscillation is generated by high-frequency input.

A more general framework is provided by the *heterogeneous multiscale method* [1] and it allows not just highly oscillatory forcing terms but also highly oscillatory coefficients, but the *quid pro quo* is that expansions can be practically generated up to $\mathcal{O}(\omega^{-r})$ only for fairly modest values of $r \geq 1$.

Story 3: The linear Schrödinger equation

The *linear Schrödinger equation*

$$\partial_t u = \mathrm{i}\varepsilon\partial_x^2 u + \mathrm{i}\varepsilon^{-1}V(x)u, \qquad t \geq 0, \quad -1 \leq x \leq 1, \tag{4.1}$$

where both the initial value $u(\,\cdot\,,0)$ and the *interaction potential* $V(x)$ are smooth and periodic, describes the quantum state of a single particle: essentially, it is the equivalent of Newton's law in a quantum setting. The parameter $\varepsilon > 0$ is very small, rendering the solution of this deceptively simple linear PDE fairly knotty.

Standard numerical wisdom goes along the following lines: we commence by discretising (4.1) in some M-dimensional space, e.g. by replacing $\partial_x^2 u$ by a linear combination of function values along a grid (finite differences) or moving to the Fourier space and considering there an L_2 projection on Nth degree trigonometric polynomials, $M = 2N + 1$ (a spectral method). The outcome is a set of linear ODEs of the form

$$\boldsymbol{u}' = \mathrm{i}(\varepsilon\mathcal{K} + \varepsilon^{-1}\mathcal{D}_V)\boldsymbol{u}, \quad t \geq 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{4.2}$$

where $\mathcal{K}$ and $\mathcal{D}_V$ are matrices corresponding to approximate differentiation and multiplication by V, respectively. Solving (4.2) should be easy in principle, after all an explicit solution is available,

$$\boldsymbol{u}(t) = \exp\left(\mathrm{i}t(\varepsilon\mathcal{K} + \varepsilon^{-1}\mathcal{D}_V)\right)\boldsymbol{u}_0, \qquad t \geq 0,$$

except that the tiny ε spoils the party. The dimension M is large (and our real interest is in considering (4.1) in a multivariate setting: the dimension soon becomes *really* large!) and the only realistic hope of computing the matrix exponential with reasonable accuracy is by *Krylov subspace methods* [6]: we replace the computation of a 'big' exponential by a (hopefully) small, $m \times m$, one, at a cost of $\mathcal{O}(m^2 M)$ operations. Except that our hope is in vain: once ε is very small, m becomes large, $m = \mathcal{O}(M)$, and the overall cost, $\mathcal{O}(M^3)$, is unacceptably large.

The good news is that a ready alternative is available: an *exponential splitting*. In its simplest manifestation, the *Strang splitting*

$$\mathrm{e}^{\mathrm{i}t(\varepsilon\mathcal{K}+\varepsilon^{-1}\mathcal{D}_V)} \approx \mathrm{e}^{\frac{1}{2}\mathrm{i}t\varepsilon\mathcal{K}}\mathrm{e}^{\mathrm{i}t\varepsilon^{-1}\mathcal{D}_V}\mathrm{e}^{\frac{1}{2}\mathrm{i}t\varepsilon\mathcal{K}} \tag{4.3}$$

incurs an error of $\mathcal{O}(t^3)$ and is easy to evaluate: e.g., once we use a spectral method, $\mathcal{K}$ is a diagonal matrix and $\mathcal{D}_V$ a circulant, whose exponential can be evaluated with two Fast Fourier Transforms (FFT) [11]. Unfortunately, the error $\mathcal{O}(t^3)$ of the Strang splitting is unacceptably large.

It is possible to design higher-order splittings, e.g.

$$\mathrm{e}^{\frac{1}{2}\alpha t\mathcal{B}}\mathrm{e}^{\alpha t\mathcal{A}}\mathrm{e}^{\frac{1}{2}(1-\alpha)t\mathcal{B}}\mathrm{e}^{(1-2\alpha)t\mathcal{A}}\mathrm{e}^{\frac{1}{2}(1-\alpha)t\mathcal{B}}\mathrm{e}^{\alpha t\mathcal{A}}\mathrm{e}^{\frac{1}{2}\alpha t\mathcal{B}}, \tag{4.4}$$

where $\mathcal{B} = \mathrm{i}\varepsilon\mathcal{K}$, $\mathcal{A} = \mathrm{i}\varepsilon^{-1}\mathcal{D}_V$ and $\alpha = (2 - \sqrt[3]{2})^{-1}$, has error of $\mathcal{O}(t^5)$. In general, we can get $\mathcal{O}(t^{2n+1})$ with $2 \cdot 3^{n-1} + 1$ terms, but this is simply much too much for practical computation with large n, even though each individual exponential is cheap. Moreover, note that all these are expansions in the step size t, but we have two other small quantities, ε and M^{-1}: a good expansion should take them all into account.

By this stage of our narrative, a remedy should stare us in the face: an asymptotic expansion! Following upon the ideas in [2], we thus assume that $t \sim \mathcal{O}\big(\varepsilon^{1/2}\big)$, $M \sim \mathcal{O}\big(\varepsilon^{-1/2}\big)$ and seek an expansion of the form

$$\mathrm{e}^{\mathcal{R}_0}\mathrm{e}^{\mathcal{R}_1}\cdots\mathrm{e}^{\mathcal{R}_s}\mathrm{e}^{\mathcal{T}_{s+1}}\mathrm{e}^{\mathcal{R}_s}\cdots\mathrm{e}^{\mathcal{R}_1}\mathrm{e}^{\mathcal{R}_0} \tag{4.5}$$

where $\mathcal{R}_0 = \mathcal{O}\bigl(\varepsilon^{-1/2}\bigr)$, $\mathcal{R}_1 = \mathcal{O}\bigl(\varepsilon^{1/2}\bigr)$, $\ldots$, $\mathcal{R}_s = \mathcal{O}\bigl(\varepsilon^{s-1/2}\bigr)$ and $\mathcal{T}_s = \mathcal{O}\bigl(\varepsilon^{s+1/2}\bigr)$: an asymptotic splitting.

An important observation with regard to (4.3–5) is that they are all *palindromic:* the same whether we read them from the left or from the right. This has a number of advantages. This time symmetry is good in minimising the number of terms because all expansions are in odd powers of the small parameter. Moreover, since both $\mathrm{i}\partial_x^2$ and multiplication by $\mathrm{i}V$ are skew-Hermitian operators, the solution operator of (4.1) is unitary and palindromy makes it easier to respect this feature under discretisation.

The right approach towards computing the asymptotic splitting (4.5) is to abandon the semi-discretisation (4.2) for the time being and split the formal solution operator, $\exp\bigl(\mathrm{i}t(\varepsilon\partial_x^2 + \varepsilon^{-1}V)\bigr)$. Only once we have done so, we replace everything with a finite-dimensional approximation. Our main weapon is the *symmetric Baker–Campbell–Hausdorff (sBCH) formula*

$$\mathrm{e}^{\frac{1}{2}X}\mathrm{e}^{Y}\mathrm{e}^{\frac{1}{2}X} = \mathrm{e}^{\mathrm{sBCH}(X,Y)}$$

where

$$\begin{aligned}\mathrm{sBCH}(tX,tY) = t(X+Y) &- t^3\bigl(\tfrac{1}{24}[[Y,X],X] + \tfrac{1}{12}[[Y,X],Y]\bigr)\\ &+ t^5\bigl(\tfrac{7}{5760}[[[[Y,X],X],X],X] + \tfrac{7}{1440}[[[[Y,X],X],X],Y]\\ &\quad + \tfrac{1}{180}[[[[Y,X],X],Y],Y] + \tfrac{1}{720}[[[[Y,X],Y],Y],Y]\\ &\quad + \tfrac{1}{480}[[[Y,X],X],[Y,X]]\bigr) + \mathcal{O}\bigl(t^7\bigr)\,.\end{aligned}$$

Note that the expansion is in odd powers of t, reaping a benefit of palindromy!

We let $\tau = \mathrm{i}\Delta t$, where Δt is the time step. The algorithm commences by setting $\mathcal{T}_0 = \tau\varepsilon^{-1}V + \tau\varepsilon\partial_x^2$ (the term we wish to split) and $\mathcal{R}_0 = \frac{1}{2}\tau\varepsilon^{-1}V$ (half the asymptotically larger term, which we wish to knock out by this stage), therefore

$$\mathrm{e}^{\tau\varepsilon^{-1}V+\tau\varepsilon\partial_x^2} = \mathrm{e}^{\mathcal{T}_0} = \mathrm{e}^{\mathcal{R}_0}\mathrm{e}^{\mathrm{sBCH}(-2\mathcal{R}_0,\mathcal{T}_0)}\mathrm{e}^{\mathcal{R}_0}.$$

Note that, having just 'decapitated' the leading term, $\mathcal{T}_1 := \mathrm{sBCH}(-2\mathcal{R}_0,\mathcal{T}_0) = \mathcal{O}\bigl(\varepsilon^{1/2}\bigr)$. We next identify the leading (i.e., $\mathcal{O}\bigl(\varepsilon^{1/2}\bigr)$) term of $\mathcal{T}_1$ and let $\mathcal{R}_1$ be half of it. Therefore

$$\mathrm{e}^{\tau\varepsilon^{-1}V+\tau\varepsilon\partial_x^2} = \mathrm{e}^{\mathcal{R}_0}\mathrm{e}^{\mathcal{R}_1}\mathrm{e}^{\mathrm{sBCH}(-2\mathcal{R}_1,\mathcal{T}_1)}\mathrm{e}^{\mathcal{R}_1}\mathrm{e}^{\mathcal{R}_0}$$

and $\mathrm{sBCH}(-2\mathcal{R}_1,\mathcal{T}_1) = \mathcal{O}\bigl(\varepsilon^{3/2}\bigr)$. We continue in this vain until the right accuracy has been attained except that, more perhaps than always, the devil is in the detail! On the face of it, the sBCH formula introduces commutators galore and this might mire the entire enterprise in expensive calculations. Except that, upon further examination, commutators go away! For example, it is easy to confirm that $[V,\partial_x^2] = -V'' - 2V'\partial_x$. In general, every combination of nested commutators of V and ∂_x^2 can be written in the form $\sum_{k=0}^{n} y_k(x)\partial_x^k$ for some $n \geq 0$ and functions $y_1,\ldots,y_n$ which depend on V and its derivatives.

Commutators go away but another problem rears its head: odd powers of ∂_x, which may cause instability and play havoc with our expansions. The remedy is to get rid of

odd derivatives by a simple trick, e.g.

$$y(x)\partial_x = -\tfrac{1}{2}\int_{x_0}^{x} y(\xi)\,\mathrm{d}\xi\partial_x^2 - \tfrac{1}{2}y'(x) + \tfrac{1}{2}\partial_x^2\left[\int_{x_0}^{x} y(\xi)\,\mathrm{d}\xi\,\cdot\right]$$

– we have only even-order derivative operators on the right and all is well! The outcome,

$$\begin{aligned}
\mathcal{R}_0 &= \tfrac{1}{2}\tau\varepsilon^{-1}V = \mathcal{O}\big(\varepsilon^{-1/2}\big)\,,\\
\mathcal{R}_1 &= \tfrac{1}{2}\tau\varepsilon\partial_x^2 + \tfrac{1}{24}\tau^3\varepsilon^{-1}V'' = \mathcal{O}\big(\varepsilon^{1/2}\big)\,,\\
\mathcal{R}_2 &= -\tfrac{1}{12}\tau^3\varepsilon\{\partial_x^2(V''\,\cdot\,) + V''\partial_x^2\} + \tfrac{1}{120}\tau^5\varepsilon^{-1}V''V^2 = \mathcal{O}\big(\varepsilon^{3/2}\big)\,,
\end{aligned}$$

etc. is our asymptotic splitting.

Now – and only now – we discretise derivatives by one of the very powerful methods of numerical analysis, e.g. spectral collocation [5]. All this procedure can be further improved by, paradoxically, first knocking out the *smaller* term, $\tau\varepsilon\partial_x^2$ since this counterintuitive step means that all $\mathcal{O}\big(\varepsilon^{m/2}\big)$ exponentials for $m \le \frac{1}{2}$ can be evaluated easily either because they are diagonal or by FFT. The remaining exponentials can be reduced by Krylov subspace methods to a *very* small number of dimensions, e.g. for $s = 2$ we need just dimension 3 for $\mathcal{R}_2$ and dimension 2 for $\mathcal{T}_3$.

Beyond univariate Schrödinger The equation (4.1) is the simplest model of linear Schrödinger equations in quantum chemistry. In general, we wish to solve the multivariate equation $\partial_t u = \mathrm{i}\varepsilon\nabla^2 u + \mathrm{i}\varepsilon^{-1}V(\boldsymbol{x})u$ in a d-dimensional torus. Provided that d is moderately small, the asymptotic expansion (4.5) generalises and, at the cost of $\mathcal{O}\big(N^d \log N\big)$ operations, remains practically feasible. However, once d is large, this approach needs to be matched by specialised methodologies for high-dimensional settings, e.g. sparse grids: this is under active research. Another generalisation is to time-dependent interaction potentials, important if magnetism effects are taken on board.

Yet, the potential scope of 'asymptotic splitting' is much wider, not just because of the asymptotic rate of decay but because commutators are replaced by easy-to-evaluate expressions. Only the future can tell the limits of this methodology.

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Recent conference announcements

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August 5 – 9, 2013, El Colegio Nacional, Mexico City

organized by
Kurt Bernardo Wolf, Octavio Novaro, Roelof Bijker, Octavio Castaños, Roco Jáuregui, Renato Lemus.

Web page http://www.fis.unam.mx/symposiaqts

This conference is partially funded by the IAMP.

Quantum Integrable Systems and Quantum Information Theory

August 18 – 24, 2013, Lawler's Hotel, Dungarvan, Ireland

organized by Tony Dorlas (STP-DIAS, Dublin, Ireland)
Alexander Stolin (Chalmers University, Gothenburg, Sweden)

The school is aimed at postgraduate students and young researchers, as well as at higher level undergraduate students with an interest in quantum systems and/or quantum information theory.

For more information on the School, including registration, please visit its webpage `http://www.dias.ie/summerschool2013` or contact Mary Brennan at DIAS, 01-6140122.

QMath12: Mathematical Results in Quantum Physics

September 10 – 13, 2013, Humboldt-Universität, Berlin, Germany

organized by
Volker Bach, Michael Demuth, Pavel Exner, Wolfgang König, Alexander Mielke, Hagen Neidhardt, Reinhold Schneider.

Web page http://www.qmath12.de

This conference is partially funded by the IAMP.

Texas Analysis and Mathematical Physics Symposium

October 25- 27, 2013, Rice University, USA

organized by
Thomas Chen and David Damanik

Web page
http://www.math.utexas.edu/users/tc/TeXAMP/TeXAMP-2013/TeXAMP-2013.html

Nonequilibrium Problems in Physics and Mathematicss

June 1 – 6, 2014, Centro Stefano Franscini, Monte Verita, Ascona, Switzerland

organized by
Jean-Pierre Eckmann (University of Geneva) and Antti Kupiainen (University of Helsinki)

Web page http://theory.physics.unige.ch/NPPM/

Open positions

Postdoctoral and doctoral positions at Stuttgart/Tübingen

Two post-doctoral position in the area of analytical quantum mechanics are available starting October 1st, 2013 at the Universities of Stuttgart and Tübingen, Germany (one position at each university). These are two-year positions within the new Research Training Group: Spectral Theory and Dynamics of Quantum Systems. For more information see http://www.mathematik.uni-stuttgart.de/grk1838.html

The deadline for applications is July 31, 2013.

Six doctoral position in the area of analytical quantum mechanics are available starting October 1st 2013 at the Universities of Stuttgart and Tübingen, Germany (three positions at each university). These positions are part of the new Research Training Group (Graduiertenkolleg): Spectral Theory and Dynamics of Quantum Systems. For more information see http://www.mathematik.uni-stuttgart.de/grk1838.html

The deadline for applications is August 31, 2013.

More job announcements are on the job announcement page of the IAMP

http://www.iamp.org/page.php?page=page_positions

which gets updated whenever new announcements come in.

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