International Association of Mathematical Physics

News Bulletin

April 2014

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Cover picture: The main building of the Institute of Science and Technology (IST) in Klosterneuburg, Austria (picture: courtesy and © IST Austria). See the article by László Erdős and Robert Seiringer in this issue.

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News Bulletin (International Association of Mathematical Physics)

The Abel Prize for 2014

The Norwegian Academy of Science and Letters has decided
to award the Abel Prize for 2014 to

Yakov G. Sinai

Princeton University and Landau Institute for Theoretical Physics,
The Russian Academy of Sciences

"for his fundamental contributions to dynamical systems, ergodic theory, and mathematical physics"

Laudatio. Ever since the time of Newton, differential equations have been used by mathematicians, scientists and engineers to explain natural phenomena and to predict how they evolve. Many equations incorporate stochastic terms to model unknown, seemingly random, factors acting upon that evolution. The range of modern applications of deterministic and stochastic evolution equations encompasses such diverse issues as planetary motion, ocean currents, physiological cycles, population dynamics, and electrical networks, to name just a few. Some of these phenomena can be foreseen with great accuracy, while others seem to evolve in a chaotic, unpredictable way. Now it has become clear that order and chaos are intimately connected: we may find chaotic behaviour in deterministic systems, and conversely, the statistical analysis of chaotic systems may lead to definite predictions. Yakov Sinai made fundamental contributions in this broad domain, discovering surprising connections between order and chaos and developing the use of probability and measure theory in the study of dynamical systems. His achievements include seminal works in ergodic theory, which studies the tendency of a system to explore all of its available states according to certain time statistics; and statistical mechanics, which explores the behaviour of systems composed of a very large number of particles, such as molecules in a gas. Sinai's first remarkable contribution, inspired by Kolmogorov, was to develop an invariant of dynamical systems. This invariant has become known as the Kolmogorov-Sinai entropy, and it has become a central notion for studying the complexity of a system through a measure-theoretical description of its trajectories. It has led to very important advances in the classification of dynamical systems. Sinai has been at the forefront of ergodic theory. He proved the first ergodicity theorems for scattering billiards in the style of Boltzmann, work he continued with Bunimovich and Chernov. He constructed Markov partitions for systems defined by iterations of Anosov diffeomorphisms, which led to a series of outstanding works showing the power of symbolic dynamics to describe various classes of mixing systems. With Ruelle and Bowen, Sinai discovered the notion of SRB measures: a rather general and distinguished invariant measure for dissipative systems with chaotic behaviour. This versatile notion has been very useful in the qualitative study of some archetypal dynamical systems as well as in the attempts to tackle real-life complex chaotic behaviour such as turbulence. Sinai's other pioneering works in mathematical physics include: random walks in a random environment (Sinai's walks), phase transitions (Pirogov-Sinai theory), one-dimensional turbulence (the statistical shock structure of the stochastic Burgers equation, by E-Khanin-Mazel-Sinai), the renormalization group theory (Bleher-Sinai), and the spectrum of discrete Schrödinger operators. Sinai has trained and influenced a generation of leading specialists in his research fields. Much of his research has become a standard toolbox for mathematical physicists. His works had and continue to have a broad and profound impact on mathematics and physics, as well as on the ever-fruitful interaction of these two fields.

Call for nominations for the 2015 IAMP Early Career Award

The IAMP Executive Committee calls for nominations for the 2015 IAMP Early Career Award. The prize was instituted in 2008 and will be awarded for the third time at the ICMP in Santiago, Chile, in July 2015.

The Early Career Award is given in recognition of a single achievement in mathematical physics. The total prize value amounts to 3000 Euro and is reserved for scientists whose age in years since birth on July 31 of the year of the Congress is less than 35.

The nomination should include the name of the candidate accompanied by a brief characterization of the work singled out for the prize. Members of the IAMP should send their nomination or nominations to the President (president@iamp.org) and to the Secretary (secretary@iamp.org). A list of previous winners and the details of the award selection process can be found at: http://www.iamp.org.

Nominations should be made not later than **January 31, 2015**.

Call for nominations for the IUPAP Young Scientist Prize in Mathematical Physics 2015

The IUPAP Mathematical Physics C18 prize (http://www.iupap.org) recognizes exceptional achievements in mathematical physics by by scientists at relatively early stages of their careers. It is awarded triennially to at most three young scientists satisfying the following criteria:

- The recipients of the awards in a given year should have a maximum of 8 years of research experience (excluding career interruptions) following their PhD on January 1 of that year, in the present case 2015.
- The recipients should have performed original work of outstanding scientific quality in mathematical physics.
- Preference may be given to young mathematical physicists from developing countries.

The awards will be presented at the ICMP in July 2015 in Santiago de Chile

Please submit your nomination to Jakob Yngvason (jakob.yngvason@univie.ac.at), Antti Kupiainen (ajkupiai@cc.helsinki.fi) and Patrick Dorey (p.e.dorey@durham.ac.uk) as officers of the IUPAP C18 Commission for Mathematical Physics.

The deadline for nominations is **August 31, 2014**.

Call for nominations for the 2015 Henri Poincaré Prize

The Henri Poincaré Prize, sponsored by the Daniel Iagolnitzer Foundation, was created in 1997 to recognize outstanding contributions in mathematical physics, and contributions which lay the groundwork for novel developments in this broad field. The prize is also created to recognize and support young people of exceptional promise who have already made outstanding contributions to the field of mathematical physics. The prize is awarded every three years at the International Congress of Mathematical Physics (ICMP), and in each case is awarded usually to three individuals.

The prize winners are chosen by the Executive Committee of the IAMP upon recommendations given by a special Prize Committee. The Executive Committee has made every effort to appoint to the prize committee prominent members of our community that are representative of the various fields it contains. However, to be able to do its job properly the Prize Committee needs input from the members of IAMP. For this purpose the Executive Committee calls IAMP members to provide nominations for the Henri Poincaré Prize to be awarded at ICMP 2015 at Santiago, Chile.

A proper nomination should include the following:

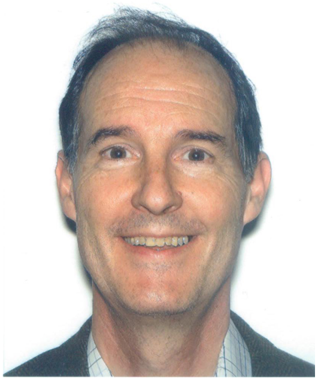
- Description of the scientific work of the nominee emphasizing their key contributions.
- A recent C.V. of the nominee.
- A proposed citation, should the nominee be selected for an award.

Please keep the length of your nomination within a page and submit it to the President (president@iamp.org) or the Secretary (secretary@iamp.org). A list of previous winners can be found at: http://www.iamp.org.

To ensure full consideration please submit your nominations by **September 30, 2014**.

Bell's inequalities, 50 years later

by T. C. Dorlas (Dublin Institute for Advanced Studies)



Teunis Christiaan Dorlas received his PhD "On Some Aspects of Renormalisation Group Theory and Hierarchical Model" from the State University of Groningen in 1987. Having held a post-doctoral position at Dublin Institute for Advanced Studies and then lecturer-reader position at University College of Swansea, he became Professor of Mathematics there in 1999. In 2000 he accepted a call from the Dublin Institute for Advanced Studies where he became Director of the School of Theoretical Physics from 2001 until 2007, and where he is currently a Senior Professor. He has established conceptual results in the mathematical theory of Bose-Einstein condensation such as a relevance of the Large Deviation Principle. He has important results concerning the Bethe Ansatz, spectral theory of random Schrödinger operators and the discrete Feynman integral theory. His recent results concern quantum information theory.

Hidden variables

John Stewart Bell was an Irish particle theorist working for the latter part of his career at CERN. (For an account of his career, see [1] and [2].) However, he is best known for his work on the foundations of quantum mechanics, which he called his 'hobby'. His most famous paper [3], in which he introduced his 'Bell inequalities' was published 50 years ago, so it seems appropriate to review these inequalities and some of their consequences at this time. In fact, his research into foundations started with another paper, which was accidentally published later. This latter paper [4] was conceived much earlier after conversations with Mandel. Bell was much impressed with the work of Bohm [5] and de Broglie [6] who developed an alternative to the standard version of quantum mechanics. Their work showed that it is possible to introduce 'hidden variables' in non-relativistic quantum mechanics which determine the quantum randomness in a way similar to statistical mechanics, i.e. random variables which cannot be observed but cause the probabilistic nature of the quantum measurement results. They suggested that the trajectory of a particle is in fact deterministic, but is under the influence of a random background force which cannot be controlled. Their theory seemed to be in conflict with a theorem by Von Neumann [7] about the impossibility of hidden variables in quantum mechanics. Of course, a mathematical theorem is based on certain assumptions, and Bell rather scathingly dismissed Von Neumann's main assumption, namely that the sum of two observables is also an observable and its expectation value is the sum of the individual expectations.

As a counter example, he introduced a simple hidden variable model for a single spin-$\frac{1}{2}$

particle, which might be worth considering here in some detail.

The algebra of observables of a spin-$\frac{1}{2}$ particle is given by

$$A = a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma},$$

where $a_0 \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. In the state $\psi_0 = |0\rangle$ (eigenstate of σ_z with eigenvalue 1), its expectation is

$$\langle a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma}\rangle = a_0 + a\cos(\alpha),$$

where α is the angle between $\vec{a}$ and the positive z-axis and $a = |\vec{a}|$. This result can also be realised by introducing a hidden variable $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform distribution, and a map

$$f_\alpha(\lambda) = \operatorname{sgn}\left(\lambda + \frac{1}{2}\cos(\alpha)\right).$$

Then

$$\langle a_0\mathbf{1} + \vec{a}\cdot\vec{\sigma}\rangle = \mathbb{E}\left[a_0 + a f_\alpha(\lambda)\right].$$

Actually, there is a more intuitive way of introducing a hidden variable, namely a uniform probability distribution on the unit sphere S^2. In that case, we put

$$f_\alpha(\theta, \phi) = \operatorname{sgn}(\theta - \alpha).$$

Then if $|\vec{a}| = 1$,

$$\langle \vec{a}\cdot\vec{\sigma}\rangle = \frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi f_\alpha(\theta,\phi)\sin(\theta).$$

A general state ψ corresponds to a unit vector $\vec{\psi}$ with angular coordinates (θ_0, ϕ_0) according to

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} e^{-i\phi_0/2}\cos(\theta_0/2) \\ e^{i\phi_0/2}\sin(\theta_0/2)\end{pmatrix}.$$

Then

$$f_{\vec{a}}(\vec{\lambda}) = \operatorname{sgn}\left(\arccos(\vec{\lambda}\cdot\vec{\psi}) - \arccos(\vec{a}\cdot\vec{\psi})\right). \tag{1}$$

This is more in line with Bell's later suggestion that the quantum state ψ should really be considered the hidden variable and the presumed hidden variable the actual state. The disturbance by the quantum state ψ then causes the measured value to differ from the actual state (here $\vec{\lambda}$, in Bohm's case the position $x(t)$). Moreover, one can introduce a time evolution as follows. A general Hamiltonian is an observable and be written as

$$H = E_0 + \vec{h}\cdot\vec{\sigma}.$$

The constant E_0 only introduces a phase in the evolution, and is therefore irrelevant. In the Heisenberg picture, the operator A above transforms according to

$$\frac{d}{dt}A(t) = i[H, A(t)] = i\sum_{i,j=1}^{3} h_i a_j(t)[\sigma_i, \sigma_j] = -2(\vec{h}\wedge\vec{a}(t))\cdot\vec{\sigma}.$$

Thus the vector $\vec{a}(t)$ rotates (precesses) in a plane perpendicular to $\vec{h}$. Now the map f becomes time-dependent:

$$f_{\vec{a}}(\vec{\lambda}, t) = \mathrm{sgn}\left(\arccos(\vec{\lambda}\cdot\vec{e}_z) - \arccos(\vec{a}(t)\cdot\vec{e}_z)\right).$$

Here the unit vector $\vec{\lambda}$ corresponds to the hidden variable point (θ, ϕ). Since $-2(\vec{h}\wedge\vec{a})\cdot\vec{\psi} = 2\vec{a}\cdot(\vec{h}\wedge\vec{\psi})$, we can also consider in the Schrödinger picture, that the state vector $\vec{\psi}(t)$ rotates (in the opposite direction) with $\vec{\psi}(0) = \vec{e}_z$. Then we write

$$f_{\vec{a}}(\vec{\lambda}, t) = \mathrm{sgn}\left(\arccos(\vec{\lambda}(t)\cdot\vec{\psi}(t)) - \arccos(\vec{a}\cdot\vec{\psi}(t)\right),$$

where $\vec{\lambda}(t)$ precesses in the same way as $\vec{u}\psi(t)$. In this picture we can therefore interpret the variable $\vec{\lambda}(t)$ as the 'real' spin rotating in a deterministic way, while the measurement result is random. This is analogous to the de Broglie-Bohm view of non-relativistic quantum mechanics of a spinless particle.

Notice that the map f_α is highly nonlinear so that Von Neumann's argument does not apply. In the mean time another argument had been put forward by Jauch and Piron [9]. They assume the logical structure axioms of quantum mechanics as they were developed by Von Neumann and Birkhoff [8]. These axioms concern yes-no measurements (corresponding to projections in quantum mechanics) and are as follows:

1. The set $\mathcal{L}$ of yes-no measurements has a 'lattice structure', i.e. there is a partial order $\leq$ on $\mathcal{L}$ such that for all $a, b \in \mathcal{L}$, there exists a least upper bound $a \cup b$ and a largest lower bound $a \cap b$.
2. For every $a \in \mathcal{L}$ there is a complement $a' \in \mathcal{L}$ such that
   (a) $(a')' = a$ for all $a \in \mathcal{L}$,
   (b) $a \cap a' = 0$ and $a \cup a' = 1$ for all $a \in \mathcal{L}$, where 0 and 1 are the trivial measurements yielding no resp. yes with certainty;
   (c) $a \leq b \implies b' \leq a'$ for all $a, b \in \mathcal{L}$.

Jauch and Piron define a **state** on $\mathcal{L}$ to be a map $p : \mathcal{L} \to [0, 1]$ such that

1. $p(0) = 0$ and $p(1) = 1$;
2. For every sequence of *disjoint* propositions $(a_n)_{n\in\mathbb{N}}$, i.e. such that $a_n \leq a'_m$ for $n \neq m$,
$$\sum_{n=1}^{\infty} p(a_n) = p\left(\bigcup_{n\in\mathbb{N}} a_n\right);$$
3. If for a sequence $(a_n)_{n\in\mathbb{N}}$, $p(a_n) = 1$ for all $n \in \mathbb{N}$, then
$$p\left(\bigcap_{n\in\mathbb{N}} a_n\right) = 1.$$

Moreover, they assume that if $a \neq b$ then there exists a state p such that $p(a) \neq p(b)$.

They define a *dispersion-free* state as a state such that $p(a) = 0$ or $p(a) = 1$ for all $a \in \mathcal{L}$, and they then say that $\mathcal{L}$ admits hidden variables if every state is an average of dispersion-free states, i.e. of the form

$$p(a) = \int_\Omega p_\lambda(a)\, \mu(d\lambda) \tag{2}$$

for some probability measure μ on Ω and a family of dispersion-free states p_λ.

Lemma. *If a proposition system $\mathcal{L}$ admits hidden variables, then for all $a, b \in \mathcal{L}$,*

$$p(a) + p(b) = p(a \cap b) + p(a \cup b).$$

Two propositions a and b are said to be **compatible** if they generate a Boolean lattice, i.e. a lattice in which the distributive law holds. With the additional assumption that if $a \leq b$ then a and b are compatible, one can show that arbitrary $a, b \in \mathcal{L}$ are compatible if and only if

$$(a \cap b') \cup b = (a' \cap b) \cup b'. \tag{3}$$

We now write

$$\begin{aligned} p((a \cap b') \cup b) &= p(a \cap b') + p(b) \\ &= p(a) + p(b') - p(a \cup b') + p(b) \\ &= p(a) + 1 - p(a \cup b') \\ &= p(a) + p(a' \cap b) = p(a \cup (a' \cap b)). \end{aligned}$$

By the assumption that the states separate the propositions, we conclude that (3) holds.

Bell objects that this is no argument for rejecting hidden variables in a wider sense. Namely, in the example above, if a and b are given by 1-dimensional projections $\frac{1}{2}(\mathbf{1} + \vec{a} \cdot \vec{\sigma})$ and $\frac{1}{2}(\mathbf{1} + \vec{b} \cdot \vec{\sigma})$, and $\vec{b} \neq \vec{a}$, then $a \cap b = 0$, so one should have $p_\lambda(a \cap b) = 0$. But in the example, $p_\lambda(a) = \frac{1}{2}(1 + f_\alpha(\lambda))$ and $p_\lambda(b) = \frac{1}{2}(1 + f_\beta(\lambda))$ both equal 1 at the same time for certain values of λ.

Bell inequalities and the EPR paradox

Einstein, Podolsky and Rosen (EPR) [10] proposed a famous 'Gedankenexperiment' to argue that quantum mechanics cannot be a complete theory. Although they used momentum and position operators for two particles, it is now usually presented in terms of spin-coordinates of two spin-$\frac{1}{2}$ particles, as suggested by Aharonov and Bohm [11]. In this formulation one considers an entangled state of the two particles, e.g. the singlet state

$$\psi_s = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

(Here the first index labels the state of one particle, the second that of the other, and $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z.) The particles can be arbitrarily far apart. Measuring the spin of one particle, e.g. with $\sigma_z \otimes \mathbf{1}$, collapses the state to $|01\rangle$ or $|10\rangle$, thus also determining the spin of the other. EPR found this problematic because in the standard interpretation of quantum mechanics the state of the other particle was indeterminate before the measurement, which seemed to imply action at a distance. They suggested that this means that quantum mechanics is incomplete: there should be 'hidden variables' which in fact determine the state of the two particles. The situation would then be analogous to a coin having been cut in half so that one half is heads, the other tails, and the two halves given to two people ('Alice' and 'Bob' in modern parlance) in closed boxes. Then once Alice opens her box, the content of Bob's box is instantaneously known. The difference is that in this case the contents of the boxes is in fact predetermined, even if they are unknown to Alice and Bob.

Bell realised that as far as measurements of the z-component of the spins is concerned, the EPR experiment is in fact classical and one cannot objectively decide about the existence of hidden variables. To do this, it is necessary to consider more general measurements. For hidden variables to be a genuine possibility, they should be able to explain more general measurements. Assuming the existence of a general hidden variable in the form of a probability measure he proceeded to derive an inequality regarding general measurements, which is not satisfied for quantum states, and hence provides a possible experimental test of the existence of hidden variables. Here we derive the slightly more general inequalities due to Clauser et al. [12]. Suppose there is a probability measure μ on a space Ω of hidden variables determining the results A and B of measurements of the spin components of the two particles in directions $\vec{a}$ and $\vec{b}$ respectively, i.e. $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$. Here the crucial assumption is that of *locality*, i.e. the outcome $A(\vec{a},\lambda)$ does not depend on $\vec{b}$ and vice-versa $B(\vec{b},\lambda)$ does not depend on $\vec{a}$. We know that each measurement results in one of the values ± 1. Consider the correlation given by

$$E(\vec{a},\vec{b}) := \int_\Omega A(\vec{a},\lambda)B(\vec{b},\lambda)\mu(d\lambda).$$

(In fact, the measuring instruments could also have hidden variables. We then need to replace A and B by averages over these instrument variables and $|A|,|B| \leq 1$ rather than $= \pm 1$.) Now, varying the instrument settings, we have

$$\begin{aligned} E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}') &= \int_\Omega A(\vec{a},\lambda)[B(\vec{b},\lambda) - B(\vec{b}',\lambda)]\mu(d\lambda) \\ &= \int_\Omega A(\vec{a},\lambda)B(\vec{b},\lambda)[1 \pm A(\vec{a}',\lambda)B(\vec{b}',\lambda)]\mu(d\lambda) \\ &\quad - \int_\Omega A(\vec{a},\lambda)B(\vec{b}',\lambda)[1 \pm A(\vec{a}',\lambda)B(\vec{b},\lambda)]\mu(d\lambda). \end{aligned}$$

Using $|A|,|B| \leq 1$, we get

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}')| \leq 2 \pm \int_\Omega [A(\vec{a}',\lambda)B(\vec{b}',\lambda) + A(\vec{a}',\lambda)B(\vec{b},\lambda)]\mu(d\lambda)$$

or

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}')| + |E(\vec{a}',\vec{b}) + E(\vec{a}',\vec{b}')| \le 2. \tag{4}$$

On the other hand, consider the quantum expectation of $AB = (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})$ in the state $\psi_0 = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. A simple calculation shows that

$$\langle \sigma_i \otimes \sigma_j \rangle = -\delta_{i,j} \text{ for } i,j = x,y,z.$$

Hence

$$\langle (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})\rangle = -\vec{a}\cdot\vec{b}. \tag{5}$$

Thus

$$\begin{aligned} &|\langle (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma}) - (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}'\cdot\vec{\sigma})\rangle| \\ &+|\langle (\vec{a}'\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma}) + (\vec{a}'\cdot\vec{\sigma})\otimes(\vec{b}'\cdot\vec{\sigma})\rangle| = |\vec{a}\cdot(\vec{b}-\vec{b}')| + |\vec{a}'\cdot(\vec{b}+\vec{b}')|. \end{aligned} \tag{6}$$

This is clearly maximal if $\vec{a}$ is in the direction of $\vec{b}-\vec{b}'$ and $\vec{a}'$ in the direction of $\vec{b}+\vec{b}'$, in which case it equals $|\vec{b}-\vec{b}'| + |\vec{b}+\vec{b}'| = \sqrt{2-2\cos\beta} + \sqrt{2+2\cos\beta}$, where β is the angle between $\vec{b}$ and $\vec{b}'$. This in turn is maximal when $\beta = \pi/2$ and the maximum value is $2\sqrt{2} > 2$. In this optimal case, therefore, the above inequality is violated.

In the mean time many experiments have confirmed with increasing confidence that the Bell inequality is not satisfied and in some cases that the quantum mechanical bound is closely approximated. Most experiments have been done with photons, see e.g. [13, 14, 16, 17, 18, 19, 20]. Notice that in order to properly test the *nonlocality*, in the above experiments the directions of polarisation analysers were changed while the photons were in flight. Initially, in [13], the measurement directions were fixed beforehand, then in [14] this was done in a periodic manner, whereas in later experiments it was done at random. Experiments have also been done with other particles, e.g. neutrons: see [21]. These experiments are very difficult and the challenge posed by Bell's inequality has thus strongly stimulated the advancement of experimental techniques.

Remark 1. It is easy to see that $\mathbb{P}[A(\vec{a}) = s, B(\vec{b}) = s']$ only depends on ss' and hence

$$\mathbb{P}[A(\vec{a}) = s,\, B(\vec{b}) = s'] = \frac{1}{4}(1 - s\,s'\vec{a}\cdot\vec{b}).$$

Thus, the measurement of $(\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})$ is essentially a measurement of the spin of one particle w.r.t. a state determined by the measurement direction of the other. Nonlocality seems quite obvious from this point of view.

Remark 2. Notice also that if we admit *signed* measures, then we can realise these probabilities as

$$\mathbb{P}[A(\vec{a}) = s,\, B(\vec{b}) = s'] = \int 1_{A(\vec{a},\lambda)=s} 1_{B(\vec{b},\lambda)=s'}\, \mu(d\lambda). \tag{7}$$

Indeed, by the above remark, it suffices if

$$\langle (\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})\rangle = \int A(\vec{a},\lambda)B(\vec{b},\lambda)\,\mu(d\lambda).$$

Let us put $\lambda = (\vec{\lambda}_1, \vec{\lambda}_2)$ and define

$$A(\vec{a}, \vec{\lambda}) = \mathrm{sgn}\,(\vec{\lambda} \cdot \vec{a}) \text{ and } B(\vec{b}, \vec{\lambda}) = \mathrm{sgn}\,(\vec{\lambda} \cdot \vec{b}).$$

Then we compute

$$\int_{S^2 \times S^2} \mathrm{sgn}\,(\vec{\lambda}_1 \cdot \vec{a})\mathrm{sgn}\,(\vec{\lambda}_2 \cdot \vec{b})\, \vec{\lambda}_1 \cdot \vec{\lambda}_2 d\vec{\lambda}_1 d\vec{\lambda}_2,$$

where $d\vec{\lambda}$ denotes normalised Lebesgue measure. This is clearly rotation-invariant, so we can take $\vec{a} = \vec{e}_z$ and $\vec{b} = \sin\gamma\, \vec{e}_x + \cos\gamma\, \vec{e}_z$. Changing variables to

$$\vec{\lambda}_2' = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \vec{\lambda}_2$$

we have

$$\begin{aligned} \vec{\lambda}_1 \cdot \vec{\lambda}_2 \quad = \quad & \cos\phi_1 \sin\theta_1 (\cos\gamma \cos\phi_2' \sin\theta_2' + \sin\gamma \cos\theta_2') \\ & + \sin\phi_1 \sin\theta_1 \sin\phi_2' \sin\theta_2' \\ & + \cos\theta_1 (-\sin\gamma \cos\phi_2' \sin\theta_2' + \cos\gamma \cos\theta_2'). \end{aligned}$$

W.r.t. these variables

$$A(\vec{a}, \vec{\lambda}_1) = \mathrm{sgn}\,(\frac{\pi}{2} - \theta_1) \text{ and } B(\vec{b}, \vec{\lambda}_2) = \mathrm{sgn}\,(\frac{\pi}{2} - \theta_2').$$

As this is independent of ϕ_1 and ϕ_2', the integrals over ϕ_1 and ϕ_2' of the terms involving $\cos\phi_1$ or $\cos\phi_2'$ vanish. Hence

$$\begin{aligned} & \int_{S^2 \times S^2} \mathrm{sgn}\,(\vec{\lambda}_1 \cdot \vec{a})\mathrm{sgn}\,(\vec{\lambda}_2 \cdot \vec{b})\, \vec{\lambda}_1 \cdot \vec{\lambda}_2\, d\vec{\lambda}_1 d\vec{\lambda}_2 \\ & = \frac{1}{4} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2' \mathrm{sgn}\,(\frac{\pi}{2} - \theta_1)\mathrm{sgn}\,(\frac{\pi}{2} - \theta_2') \cos\gamma \cos\theta_1 \cos\theta_2' \sin\theta_1 \sin\theta_2' \\ & = \frac{1}{4} \cos\gamma. \end{aligned}$$

The measure

$$\mu(d\lambda) = -4\vec{\lambda}_1 \cdot \vec{\lambda}_2\, d\vec{\lambda}_1 d\vec{\lambda}_2$$

therefore satisfies (7).

This is somewhat reminiscent of Feynman's integral, which is also a complex-valued measure in the finite-dimensional case: see [22].

Quantum information

Entanglement

Of course, the essential feature of the singlet state ψ_s is that it is *entangled*: it cannot be written as a tensor product. This crucial feature of general quantum states was highlighted (and named) by Schrödinger in two papers, one in German [23] and one in English [24], in reaction to the EPR paper. He reasoned that entanglement is in fact the crucial distinguishing feature of quantum mechanics and is also at the root of the nature of measurement. In order to illustrate the absurdity of the situation, he introduced his famous cat.

In fact, it is easy to see that any entangled state violates the Bell inequality. Namely, an arbitrary state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be written in the form

$$\psi = \lambda_1 |0\rangle \otimes |0\rangle' + \lambda_2 |1\rangle \otimes |1\rangle',$$

where $\lambda_1, \lambda_2 \geq 0$, $\lambda_1^2 + \lambda_2^2 = 1$ and $|0\rangle, |1\rangle$ and $|0\rangle', |1\rangle'$ are orthogonal bases. This is obviously entangled unless $\lambda_1 \lambda_2 = 0$. Considering the expectation value

$$E(\vec{a}, \vec{b}) = \langle \psi \,|\, (\vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma}' \,|\, \psi \rangle,$$

where $\vec{\sigma}'$ represent the Pauli matrices on the basis $\{|0\rangle', |1\rangle'\}$, we have

$$\begin{aligned} &|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| + |E(\vec{a}', \vec{b}) + E(\vec{a}', \vec{b}')| \\ &= |a_z(b_z - b'_z) + 2\lambda_1\lambda_2(a_x(b_x - b'_x) - a_y(b_y - b'_y))| \\ &+ |a_z(b_z + b'_z) + 2\lambda_1\lambda_2(a_x(b_x + b'_x) - a_y(b_y + b'_y))|. \end{aligned}$$

Maximising over $\vec{a}$ and $\vec{a}'$ we get

$$\begin{aligned} &\sqrt{(b_z - b'_z)^2 + 4\lambda_1^2\lambda_2^2((b_x - b'_x)^2 + (b_y - b'_y)^2)} \\ &+ \sqrt{(b_z + b'_z)^2 + 4\lambda_1^2\lambda_2^2((b_x + b'_x)^2 + (b_y + b'_y)^2)}. \end{aligned}$$

Taking for example $\vec{b} = \vec{e}_x$ and $\vec{b}' = \vec{e}_z$, this is $2\sqrt{1 + 4\lambda_1^2\lambda_2^2} > 2$ unless $\lambda_1\lambda_2 = 0$.

In general, quantum systems are in a mixed state, so it is interesting to wonder to what extent Bell inequalities are satisfied for mixed states. *Mixed states* are given by density matrices ρ, i.e. non-negative matrices with trace equal 1. A natural generalisation of an entangled mixed state is a non-separable state: A state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called *separable* if it can be written as a convex combination of product states,

$$\rho = \sum_{i=1}^m c_i \rho_i^{(1)} \otimes \rho_i^{(2)}. \tag{8}$$

Indeed, it is easy to see that such states admit a hidden-variable model for the correlations between A and B and Bell's inequalities hold. However, it was discovered by Werner [25] that there exist non-separable states which nevertheless satisfy Bell's inequalities and even admit a classical (hidden variable) model. His example is as follows:

$$\rho_W = \frac{1}{6} \begin{pmatrix} 1+q & 0 & 0 & 0 \\ 0 & 2-q & 2q-1 & 0 \\ 0 & 2q-1 & 2-q & 0 \\ 0 & 0 & 0 & 1+q \end{pmatrix}, \tag{9}$$

where $q \in [-1, 1]$. (In fact, his construction is valid for higher dimensions, which is relevant in connection with an argument using Gleason's theorem demonstrating the impossibility of hidden variables, which is only valid for $d > 2$.) Now, $q = \mathrm{Tr}\,(V\rho_W)$, where V is the exchange operator

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that if ρ is of the form (8) then $\mathrm{Tr}\,(\rho\, V) \geq 0$, so ρ_W is not separable if $q < 0$.

A hidden variable model for this state in the form

$$\int_\Omega f_A(a, \lambda)\, f_B(b, \lambda)\, \mu(d\lambda) = \mathrm{Tr}\,(\rho_W\, P_a \otimes Q_b), \tag{10}$$

where P_a is the eigenprojection of A for the eigenvalue a, and similarly Q_b for B, can be constructed as follows. We can assume that A and B are 1-dimensional projections, $A = \frac{1}{2}(\mathbf{1} + \vec{a}\cdot\vec{\sigma})$ and $B = \frac{1}{2}(\mathbf{1} + \vec{b}\cdot\vec{\sigma})$, and take the measure space Ω to be the unit sphere S^2 with normalised Lebesgue measure as before, and define

$$f_A(\vec{\lambda}) = \mathrm{Tr}\,(A\, P_{\vec{\lambda}})$$

and

$$f_B(\vec{\lambda}) = 1_{\{\vec{\lambda}:\, \vec{b}\cdot\vec{\lambda} < 0\}}.$$

Then the left-hand side of (10) equals

$$\int_{S^2} \frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda})\, 1_{\{\vec{\lambda}:\, \vec{b}\cdot\vec{\lambda} < 0\}} d\vec{\lambda}.$$

To compute this, we may assume $\vec{b} = \vec{e}_z$ and $\vec{a}$ given by polar angles (θ_a, ϕ_a) and $\vec{a}\cdot\vec{\lambda} = \cos\theta_a \cos\theta + \cos(\phi_a - \phi)\sin\theta_a \sin\theta$, so

$$\begin{aligned}
&\int_{S^2} \frac{1}{2}(1 + \vec{a}\cdot\vec{\lambda})\, 1_{\{\vec{\lambda}:\, \vec{b}\cdot\vec{\lambda} < 0\}} d\vec{\lambda} \\
&= \frac{1}{4\pi} \int_{\pi/2}^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \frac{1}{2} \{1 + \cos\theta_a \cos\theta + \cos(\phi_a - \phi)\sin\theta_a \sin\theta\} \\
&= \frac{1}{4} - \frac{1}{8}\cos\theta_a.
\end{aligned}$$

On the other hand, the right-hand side of (10) is

$$\mathrm{Tr}\, \left[\rho_W \frac{1}{2}(\mathbf{1} + \vec{a}\cdot\vec{\sigma}) \otimes \frac{1}{2}(\mathbf{1} + \vec{b}\cdot\vec{\sigma})\right] = \frac{1}{4}\left(1 + \frac{2}{3}(q - \frac{1}{2})\vec{a}\cdot\vec{b}\right).$$

It follows that $q = -\frac{1}{4}$ and the state is not separable. Since it admits a classical model (10) Bell's inequalities (and generalisations) are satisfied, and these inequalities are therefore insufficient to conclude that a state is classically correlated (separable).

Remark. This representation is not entirely satisfactory since the function $f_A(\lambda)$ is not an indicator function, i.e. it does not take values in $\sigma(A)$. However, this can be remedied by writing $\frac{1}{2}(1 + \vec{a} \cdot \vec{\lambda})$ in the form (1):

$$\frac{1}{2}(1 + \vec{a} \cdot \vec{\lambda}) = \int 1_{\{\vec{\lambda}' : (\vec{\lambda}' - \vec{\lambda}) \cdot \vec{a} < 0\}} d\vec{\lambda}'.$$

Moreover, replacing f_B by

$$f_B(\vec{\lambda}) = 1_{\{\vec{\lambda} : u < \vec{b} \cdot \vec{\lambda} < u+1\}}$$

we obtain

$$\int_{S^2} \frac{1}{2}(1 + \vec{a} \cdot \vec{\lambda})\, 1_{\{\vec{\lambda} : \vec{b} \cdot \vec{\lambda} \in (u, u+1)\}} d\vec{\lambda}$$
$$= \frac{1}{4}(1 + (u + \frac{1}{2}) \cos \theta_a).$$

This covers the range $q \in [-\frac{1}{4}, 0]$ when $u \in [-\frac{1}{2}, -1]$.

In fact, in the case of a pair of spin-$\frac{1}{2}$ particles, a necessary and sufficient condition for a state to be of the form (8) was introduced by Peres [26]. Introducing the *partial transpose* ρ^{T2} by

$$\langle ik \,|\, \rho^{T2} \,|\, jl \rangle = \langle il \,|\, \rho \,|\, jk \rangle, \tag{11}$$

we say that ρ is positive under partial transposition if ρ^{T2} is also a positive definite matrix. It is clear that this is a necessary condition for a state to be separable, i.e. of the form (8). It was shown by Horodecki et al. [27] that for the case of spin-$\frac{1}{2}$ particles, it is also sufficient. However, this is not so for higher-dimensional cases.

Quantum teleportation

It is nowadays recognised that entanglement can in fact be a useful resource for quantum operations. An example of this is quantum teleportation. This is a scheme for moving a quantum state from one place to another using a shared entangled state, but transmitting only classical information. It assumes that quantum states can be accurately and reliably manipulated, i.e. it is possible to apply well-defined unitary evolutions. The original example due to Bennett et al. [28] is as follows:

Assume that Alice wants to send a general qubit state $\psi = \alpha|0\rangle + \beta|1\rangle$ to Bob, and they each possess one half of a singlet state ψ_s. Alice first performs a CNOT operation on ψ and her half of the Bell state, i.e. the unitary

$$U_{CN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The resulting combined state is:

$$U_{CN}\psi \otimes \psi_s = \frac{1}{\sqrt{2}} \left(\alpha|0\rangle \otimes (|01\rangle - |10\rangle) + \beta|1\rangle \otimes (|11\rangle - |00\rangle)\right).$$

Next she applies a Hadamard operation to the first qubit: $U_H \otimes \mathbf{1}$, with

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

This yields

$$\begin{aligned} & \frac{1}{2} \left(\alpha(|0\rangle \otimes +|1\rangle) \otimes (|01\rangle - |10\rangle)\right. \\ & \left.+ \beta(|0\rangle - |1\rangle) \otimes (|11\rangle - |00\rangle)\right) && (12) \\ = \; & \frac{1}{2} \left(|00\rangle \otimes (\alpha|1\rangle - \beta|0\rangle) - |01\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)\right. \\ & \left.+ |10\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) - |11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)\right). && (13) \end{aligned}$$

Finally she performs a measurement on her parts of the combined state resulting in one of the terms in brackets of (12). If her measurement results in $(1, 1)$ then Bob's state is just ψ. Otherwise, she needs to transmit her measurement result to Bob, who can then perform a suitable unitary transformation himself to bring the state back to ψ. For example, if the result is $(1, 0)$ the third term results and he needs to act with σ_x.

Notice that the inverse operation $U_{CN}(U_H \otimes \mathbf{1})$ maps the standard basis to the basis $\{\psi_k\}_{k=0}^3$ consisting of 'Bell states'

$$\begin{aligned} \psi_{0,1} &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\ \psi_{2,3} &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \end{aligned}$$

One can therefore also say that Alice simply performs a measurement w.r.t. this basis.

One can wonder if teleportation of states is possible using more general entangled states between Alice and Bob. In that case, the teleportation is obviously not going to be perfect. As for general quantum channels, one therefore introduces the concept of *fidelity* of transmission. If Alice wants to transmit a (pure) state ψ to Bob, but the state received by Bob is the (mixed) state ν, then the fidelity is defined by the overlap

$$\mathcal{F}_\psi = \langle \psi \,|\, \nu \,|\, \psi \rangle = \mathrm{Tr}\,(\nu\, P_\psi).$$

Now, suppose Alice performs a measurement w.r.t. the Bell basis $\{\psi_k\}_{k=0}^3$, obtaining one of the results $k = 0, 1, 2, 3$ with probability p_k. She sends this result to Bob, as above, who performs a unitary transformation U_k to obtain the state ν_k. The expected value of the fidelity is then

$$\mathbb{E}(\mathcal{F}_\psi) = \sum_{k=0}^{3} p_k \,\mathrm{Tr}\,(\nu_k\, P_\psi).$$

A measure of the efficiency of this procedure is given by the average of this quantity over possible states ψ:

$$\overline{\mathcal{F}} = \int \mathbb{E}(\mathcal{F}_\psi) d\psi = \sum_{k=0}^{3} p_k \int \mathrm{Tr}\,(\nu_k\, P_\psi)\, d\psi. \tag{14}$$

If ρ is the shared entangled state and $P_k = |\psi_k\rangle\langle\psi_k|$ $(k = 0, 1, 2, 3)$ are the projections corresponding to the measurement basis, then Bob's output state ν_k is

$$\nu_k = \frac{1}{p_k}\,\mathrm{Tr}_{\;1,2}\,[(P_k \otimes U_k)(P_\psi \otimes \rho)(P_k \otimes U_k^*)]$$

and the probabilities p_k are

$$p_k = \mathrm{Tr}\,[(P_k \otimes \mathbf{1})(P_\psi \otimes \rho)].$$

Following Horodecki et al. [29], we write ρ in terms of the basis of Pauli matrices:

$$\rho = \frac{1}{4}\left(\mathbf{1} + \vec{r}\cdot\vec{\sigma}\otimes\mathbf{1} + \mathbf{1}\otimes(\vec{s}\cdot\vec{\sigma}) + \sum_{n,m=1}^{3} t_{nm}\sigma_n\otimes\sigma_m\right). \tag{15}$$

Putting $P_\psi = \frac{1}{2}(\mathbf{1} + \vec{a}\cdot\vec{\sigma})$, we have

$$\begin{aligned} p_k\nu_k \;=\; & \frac{1}{8}\Big\{ \mathrm{Tr}\,[P_k(\mathbf{1}+\vec{a}\cdot\vec{\sigma})\otimes(\mathbf{1}+\vec{r}\cdot\vec{\sigma})P_k]\,\mathbf{1} \\ & + \mathrm{Tr}\,[P_k((\mathbf{1}+\vec{a}\cdot\vec{\sigma})\otimes\mathbf{1})P_k]\,U_k(\vec{s}\cdot\vec{\sigma})U_k^* \\ & + \sum_{n,m=1}^{3} t_{nm}\,\mathrm{Tr}\,[P_k((\mathbf{1}+\vec{a}\cdot\vec{\sigma})\otimes\sigma_n)P_k]\,U_k\sigma_m U_k^* \Big\}. \end{aligned}$$

We now use

$$\begin{aligned} \langle\psi_{0,1}\,|\,\sigma_n\otimes\sigma_m\,|\,\psi_{0,1}\rangle &= \pm\delta_{n,1}\delta_{m,1} \mp \delta_{n,2}\delta_{m,2} + \delta_{n,3}\delta_{m,3} \\ \langle\psi_{2,3}\,|\,\sigma_n\otimes\sigma_m\,|\,\psi_{2,3}\rangle &= \pm\delta_{n,1}\delta_{m,1} \pm \delta_{n,2}\delta_{m,2} - \delta_{n,3}\delta_{m,3}. \end{aligned}$$

The result is:

$$p_k\nu_k = \frac{1}{8}\left\{(1 + \vec{a}\cdot D_k\vec{r})\,\mathbf{1} + U_k\left((\vec{s}\cdot\vec{\sigma}) + \vec{a}\cdot D_k T\vec{\sigma}\right)U_k^*\right\}, \tag{16}$$

where D_k are diagonal matrices: $D_0 = \mathrm{diag}(+1,-1,+1)$, $D_1 = \mathrm{diag}(-1,+1,+1)$, $D_2 = \mathrm{diag}(+1,+1,-1)$, $D_3 = \mathrm{diag}(-1,-1,-1)$. The unitary transformation U_k affects a rotation of the vector $\vec{s}$:

$$U_k(\vec{s}\cdot\vec{\sigma})U_k^* = (O_k\vec{s})\cdot\vec{\sigma}.$$

Averaging over ψ according to the uniform measure over $\vec{a}\in S^2$, we have

$$\int_{S^2}(\vec{a}\cdot A\vec{a})\,d\vec{a} = \frac{1}{3}\,\mathrm{Tr}\;A$$

and taking the trace using $\mathrm{Tr}\ \sigma_i = 0$, we get

$$\overline{\mathcal{F}} = \frac{1}{8}\sum_{k=0}^{3} p_k \left(1 + \frac{1}{3}\,\mathrm{Tr}\ D_k T O_k\right). \tag{17}$$

We need to maximise this expression over all possible choices of U_k, or equivalently O_k $(k = 0, 1, 2, 3)$. Since each D_k is a reflection, each term has the same maximum and

$$\max_{\{U_k\}} \overline{\mathcal{F}} = \max_O \frac{1}{2}(1 - \frac{1}{3}\,\mathrm{Tr}\ TO).$$

Note that if $\rho = \rho^{(1)} \otimes \rho^{(2)}$ is a product state and we write $\rho^{(1)} = \frac{1}{2}(\mathbf{1} + \vec{r}\cdot\vec{\sigma})$, $\rho^{(2)} = \frac{1}{2}(\mathbf{1} + \vec{s}\cdot\vec{\sigma})$, where in general $|\vec{r}| \le 1$ and $|\vec{s}| \le 1$, then $t_{nm} = r_n s_m$ and hence

$$\overline{\mathcal{F}} = \max_O \frac{1}{2}\left(1 - \frac{1}{3}\langle \vec{s}, O\vec{r}\rangle\right) = \frac{1}{2}\left(1 + \frac{1}{3}|\vec{r}|\,|\vec{s}|\right) \le \frac{2}{3}.$$

For separable states, therefore, the maximum is $2/3$, attained for a pure product state.

In order that a general entangled state ρ improves on this, we need $\mathrm{Tr}\,(TO) < -1$. This is the case if $\det(T) < 0$ and $||T||_1 > 1$ because in that case we can define O by $-T\psi \mapsto |T|\psi$.

Horodecki et al. [30] show that the states ρ can be written as $\rho = (U_1 \otimes U_2)\tilde{\rho}(U_1 \otimes U_2)^*$, where $\tilde{\rho}$ has a diagonal matrix T belonging to the tetrahedron with corners

$$\vec{t_0} = (-1, -1, -1),\ \vec{t_1} = (-1, 1, 1),\ \vec{t_2} = (1, -1, 1),\ \vec{t_3} = (1, 1, -1).$$

It follows from this that $||T||_1 > 1$ is in fact a necessary and sufficient condition. Moreover, they also show that the diagonal matrices for separable states belong to the octahedron with corners

$$\vec{o}_1^{\pm} = (\pm 1, 0, 0),\ \vec{o}_2^{\pm} = (0, \pm 1, 0) \text{ and } \vec{o}_3^{\pm} = (0, 0, \pm 1).$$

This implies that all non-separable states are useful for state teleportation in the sense that $\overline{\mathcal{F}} > 2/3$. For example, for Werner's state, which can be written as

$$\rho_W = \frac{1}{4}\left(\mathbf{1} + \frac{2}{3}(q - \frac{1}{2})\sum_{i=1}^{3} \sigma_i \otimes \sigma_i\right),$$

$||T||_1 = |2q - 1| > 1$ for all $q \in [-1, 0)$, i.e. whenever the state is not separable, even if Bell's inequalities hold. This was first remarked by Popescu [31].

Quantum channels

State teleportation is a special example of a quantum channel. Information is transmitted in the form of quantum states. This can be classical information (bits) or quantum information (quantum states or qubits). (In the case of teleportation, the shared quantum state can be seen to be the channel, but in addition classical side-information is transmitted.)

Information theory, initiated by Shannon, has largely been extended to the quantum domain. In particular, there is an analogue of Shannon's theorem about the capacity of a channel [32], both for the case of classical information and for quantum information. For classical information, the result is due to Holevo and Schumacher and Westmoreland [45, 33]. (See also [35] and [36].) A quantum channel can be modelled by a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ mapping (in general mixed) states on the input Hilbert space $\mathcal{H}$ to states on the output Hilbert space $\mathcal{K}$. In the case of a *memoryless channel* this map acts repeatedly, and a classical message is encoded by Alice into a quantum state $\rho^{(n)}$ of $\mathcal{H}^{\otimes n}$. The output state $\sigma^{(n)} = \Phi^{\otimes n}(\rho^{(n)})$ is then decoded by Bob by performing a generalised measurement. Such a measurement is given by a set of positive operators (not necessarily projections) $\{E_j^{(n)}\}$ with $\sum_j E_j^{(n)} = \mathbf{1}$. This is called a positive-operator-valued measure (POVM). The probability of outcome j is then given by $\mathrm{Tr}\,(\sigma^{(n)} E_j^{(n)})$. As in the case of Shannon's theorem, the (classical) *capacity* of the channel is then given by the maximal rate

$$\lim_{n\to\infty} \frac{1}{n} \log_2 N_n$$

at which messages can be transmitted with negligible error in the limit as $n \to \infty$. More precisely, one has:

Theorem. *Given $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ there are at least $N_n = [2^{n(\chi(\Phi)-\epsilon)}]$ product states $\rho_1^{(n)}, \dots, \rho_{N_n}^{(n)} \in \mathcal{B}(\mathcal{H}^{\otimes n})$ and a POVM $\{E_j^{(n)}\}_{j=1}^{N_n}$ such that $\mathrm{Tr}\left(\Phi^{\otimes n}(\rho_j^{(n)}) E_j^{(n)}\right) > 1 - \epsilon$ for all j.*

Here the quantity $\chi(\Phi)$ is the *Holevo capacity* given by

$$\chi(\Phi) = \sup_{\{\rho_j, p_j\}} \left[S\left(\sum_j p_j \Phi(\rho_j)\right) - \sum_j S(\Phi(\rho_j)) \right],$$

where S is the Von Neumann entropy $S(\rho) = -\mathrm{Tr}\,(\rho \log \rho)$, and the supremum is over ensembles of states $\rho_j \in \mathcal{B}(\mathcal{H})$ with probabilities p_j. If general states $\rho_j^{(n)}$ are admitted, the capacity is a limit:

$$\lim_{n\to\infty} \frac{1}{n} \chi(\Phi^{\otimes n})$$

but this quantity is obviously not easily computed. There are also extensions to channels with memory: see [37] and [38].

The quantum analogue of this theorem was proved by Devetak [39]. Here, one encodes and decodes states according to $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_P^{\otimes n})$ and $\mathcal{D} : \mathcal{B}(\mathcal{H}_Q^{\otimes n}) \to \mathcal{B}(\mathcal{H})$ and one wants to transmit an arbitrary (pure) state $\phi \in \mathcal{H}$ with near-perfect fidelity:

$$\min_{\phi \in \mathcal{H}} \mathcal{F}\left(\phi, (\mathcal{D} \circ \Phi^{\otimes n} \circ \mathcal{E})(|\phi\rangle\langle\phi|)\right) > 1 - \epsilon. \tag{18}$$

The analogue of Holevo's quantity is the *coherent information* $I_c(\rho, \Phi)$. It is given by

$$I_c(\rho, \Phi) = S(\Phi(\rho)) - S(\rho, \Phi)$$

where $S(\rho, \Phi)$ is the entropy exchange: see [40]. His theorem then reads as follows:

Theorem. *Given $\epsilon > 0$ there exists n_0 such that if $N_n = [2^{n(I(\Phi)-\epsilon)}]$, where*

$$I(\Phi) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^{(n)}\in\mathcal{H}_P^{\otimes n}} I_c(\rho^{(n)}, \Phi^{\otimes n}),$$

then for a Hilbert space $\mathcal{H}^{(n)}$ of dimension N_n there are encoding and decoding maps $\mathcal{E}$ and $\mathcal{D}$ such that (18) holds.

Quantum field theory

It is worth mentioning a generalisation of the Bell inequalities to quantum field theories considered by Summers and Werner [41, 42]. If $\mathcal{A}$ and $\mathcal{B}$ are commuting sub C*-algebras of a C* algebra $\mathcal{C}$ and ω is a state on $\mathcal{C}$, then whenever $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ satisfy $-\mathbf{1}_{\mathcal{A}} \le A_i \le \mathbf{1}_{\mathcal{A}}$ and $-\mathbf{1}_{\mathcal{B}} \le B_j \le \mathbf{1}_{\mathcal{B}}$ then

$$\chi := \frac{1}{2}|\omega(A_1(B_1 + B_2)) + \omega(A_2(B_1 - B_2))| \le \sqrt{2}.$$

Moreover, if ω is separable then $\chi \le 1$.

In the algebraic framework of relativistic quantum field theory, $\mathcal{A}$ and $\mathcal{B}$ can be local algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ where O_1 and O_2 are space-like separated. Assuming in particular that there is a unitary representation U of the translation group which acts covariantly, i.e.

$$U(x)\mathcal{A}(O)U(x)^{-1} = \mathcal{A}(O_x) \text{ for } x \in \mathbb{R}^4$$

and a unique vacuum vector Ω, the corresponding state ϕ_0 given by $\phi_0(A) = \langle\Omega, | A\Omega\rangle$ satisfies a much more stringent bound:

$$\chi \le 1 + 2e^{-md(O_1,O_2)},$$

where $d(O_1, O_2)$ is the maximal time-like distance between O_1 and O_2, and it is assumed that the Hamiltonian H has spectrum contained in $\{0\} \cup [m, +\infty)$ with $m > 0$. This suggests that verifying the violation of Bell's inequality is unrealistic in massive field theories. They also show, however, that in case O_1 and O_2 are complimentary 'wedges', Bell's inequality is generically maximally violated in quantum field theories, more precisely, χ approaches $\sqrt{2}$ for suitable sequences of observables with norm ≤ 1.

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The Legacy of Vladimir Andreevich Steklov in Mathematical Physics: Work and School

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The 150th anniversary of the birth of the outstanding Russian mathematician Vladimir Andreevich Steklov fell on 9 January 2014. All over the world, active researchers in all areas of mathematics know his name. Indeed, well-known mathematical institutes of the Russian Academy of Sciences in Moscow and St. Petersburg are named after Steklov. This commemorates that he was the founding father of their predecessor: the Physical–Mathematical Institute, established in 1921 in starving Petrograd (the Civil War still persisted in some corners of what would become the USSR the next year). Steklov was the first director of the institute, until his untimely death on 30 May 1926.

Meanwhile, Steklov's personality is less known even in present-day Russia. Of course, the biographical sketch [39] by J. J. O'Connor and E. F. Robertson is available online but still the best source of information about Steklov and his work is the very rare book [16]: the proceedings of a session of the Leningrad Physical–Mathematical Society, which took place on the occasion of the first anniversary of Steklov's death. It must be said that the lack of knowledge about his work was the reason for translating into Russian and publishing a collection of Steklov's papers concerning various problems in mechanics [62]. Not too much has been written about his relationship with a group of bright students (most of them graduating from St. Petersburg University in 1910). Together with students of Andrey Andreevich Markov Sr. and Nikolay Maksimovich Günther, they formed the germ which would later develop into the Petrograd–Leningrad–Petersburg school, famous for contributions of its scholars to mathematical physics, functional analysis and some other areas of mathematics as well as theoretical physics.

To clarify the word "school", which has various meanings in Russian as well as in English, it is worth quoting A. N. Parshin's recent note [40].

> A school is a community of individuals who work in the same branch of science, who are in close communication with each other, who have a leader, a teacher, amongst whom each generation passes on the torch to the next one, and all this forms one integral organism.

After this "definition", Parshin describes the branching at the Mekh-Mat (the Faculty of Mechanics and Mathematics) of the Moscow University from the original school founded by N. N. Luzin.

Speaking about Steklov's school, founded a little bit earlier than Luzin's, I understand it in the same sense as the latter is treated by Parshin. Of course, after a lapse of 100 years, the school of which Steklov was the founding father has given rise to various schools in the widely understood field of mathematical physics. It is worth mentioning that his school very soon became international. Indeed, J. D. Tamarkin and J. A. Shohat (they were members of Steklov's circle of students) emigrated in the 1920s to the USA, where they had many PhD students. In particular, Tamarkin had 28 students and so his 1495 descendants (the Mathematics Genealogy Project as of 22 August 2013) are Steklov's "scientific" grandchildren, great-grandchildren and so on. At the same time, the project gives only 888 as his own descendants.

Of course, Steklov's interests in science were much wider than mathematical physics; for example, the above mentioned collection [62] includes 12 of his major contributions to mechanics (440 pages in total). An idea of development of Steklov's work in mechanics during the past decade can be obtained from the papers [5], [66] and references cited therein. Another area which he studied during 26 years is the theory of orthogonal polynomials. In the reference book [46], one finds 31 papers by Steklov (the first two published in 1900 and the last two dated 1926) and the properties of polynomials investigated in each of them are clearly indicated. Fortunately, this topic of Steklov's research is covered in the survey article [63]; further progress can be found in [43] and [44].

This paper consists of three sections. In the first one, I briefly describe the non-scientific legacy of Steklov and then turn to quoting his own writings taken from various sources. These excerpts describe his personality and the way he started creating his school. The latter involves his relations with J. D. Tamarkin, A. A. Friedmann, V. I. Smirnov, J. A. Shohat and his other talented students shown through the prism of Steklov's diaries and recollections. Then his activity as Vice-President of the Russian Academy of Sciences during the last seven years of his life is presented in the same way. Steklov's role was crucial for the academy's survival during the period of revolutions and the Civil War in Russia. He exemplified how to withstand governmental attacks on the academy, something that is particularly important nowadays.

Since Steklov's major achievements in mathematical physics have been summarised in his book [57] entitled *Fundamental Problems of Mathematical Physics*, its contents and significance are discussed in the second section, whereas advances in the area of the potential-theoretic approach to boundary value problems for the Laplace equation (the topic of the book's volume 2) are described in the brief third section.

It should be mentioned that many of Steklov's papers are now available online as well as a complete list of his publications. For the latter see http://steklov150.mi.ras.ru/steklov_pub.pdf. The journals *Annales Fac. Sci. Toulouse* and *Annales Sci. ENS*, in which many of Steklov's articles are published, are available at http://www.numdam.org. Almost 20 of his papers in French and some papers in Russian can be found at http://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=27728.

V. A. Steklov about himself, his students, science and the Academy of Science

The legacy of Steklov is multifaceted; along with his work in mathematics and mechanics (see [68] for a survey), it includes scientific biographies of Lomonosov and Galileo, an essay about the role of mathematics (these three books in Russian were printed in 1923 in Berlin because Russian economics was ruined during World War I and the Civil War), the travelogue of his trip to Canada, where he participated in the Toronto ICM in 1924, his correspondence — published (see [60] and [61]) and unpublished — recollections [59] and still unpublished diaries.

Fortunately, many excerpts from Steklov's diaries are quoted in [65] (some of them appeared in [39] as well). In my opinion, the most expressive is dated 2 September 1914, one month after war with Germany and Austria-Hungary was declared by the Russian government.

> St. Petersburg has been renamed Petrograd by Imperial Order. Such trifles are all our tyrants can do — religious processions and extermination of the Russian people by all possible means. Bastards! Well, just you wait. They will get it hot one day!

What happened in Russia over several years after that confirms clearly how right Steklov was in his assessment of the Tsarist regime. In his recollections [59] written in 1923, he describes vividly and, at the same time, critically "the complete bacchanalia of power" preceding the collapse of "autocracy and [Romanov's] dynasty" in February 1917 (old style), "the shameful transient government headed by Kerensky, the fast end of which can be predicted by every sane person", how "the Bolshevik government [. . .] decided to accomplish the most Utopian socialistic ideas in multi-million Russia"; the list can easily be continued. My aim is to give quotations from [59] and from the unpublished manuscript *Excerpts from my diaries* (*Excerpts* in what follows) widely quoted in [22], that characterise Steklov's personality, his relations with a group of talented students at the St. Petersburg University, his understanding of the role of science for himself and his work as Vice-President of the Academy of Sciences. Translation from the Russian is mine, if not stated otherwise.

Years of education. In the *Excerpts,* Steklov writes about his final years at Alexander Institute (a kind of gymnasium) in Nizhni Novgorod.

> I turned my room into a kind of "physical cabinet" — laboratory equipped with Leyden jars, an electrical machine and home-made Galvanic elements. Various chemical experiments (of course, elementary) were carried out. [...] I reduced my contacts with schoolmates (previously numerous), continuing to keep in touch only with those of them who, like me, were interested in mathematics and physics.
>
> ... ...
>
> The topic of "test composition" [preceding the certificate exam] was as follows: *The reign of Catherine II was a great period.* In a [satirical] poem [by A. K. Tolstoy published not long before that], there are several lines characterising her in a way far from being respectful to "Her Majesty". However, they added a specific colouring to my essay. [...] I wrote, without any idea to manifest political freethinking, that Catherine's period only looks great but, in fact, it puts an end to reforms initiated by Peter I. [...] To my great surprise, Shaposhnikov (Director of the Institute) came to the classroom after reading our essays [...] and asked me: "Where have you, our best student, got this inclination to freethinking[1] and such an impermissible attitude toward the Great Empress?" [...] For almost an hour, he was explaining my thoughtlessness and my incorrect understanding of history, etc. [...] After that, he dragged in by the head and shoulders the following point of view: preferring mathematics, physics and chemistry to other disciplines, I follow an objectionable way. He said: "Maybe this is the reason that you 'took those liberties in thinking'. This feature of yours has been noted long before but definitely revealed itself in your essay."
>
> I repeat that it was a surprise for me, but did not become a stimulus to change my mind. [...] Just the opposite, I said to myself: "Aha! It occurs that I have my own point of view on historical events and it is different from that of my schoolmates and teachers. [...] It was the director himself who proved that I am, in some sense, a self-maintained thinker and critic." This was the initial impact that led to my mental awakening; I realised that I am

[01] There are two words meaning freethinking in Russian. One of them — "vol'nodumstvo" — has a negative nuance and it was used by the director.

> a human being able to reason and what is important is to reason freely. [...] Soon after that, my freethinking encompassed religion as well. [...] Thus, the cornerstone was laid to my future complete lack of faith.

In another passage from the *Excerpts*, Steklov describes how he failed to pass an examination at Moscow University.

> The last oral exam was in physical geography taught by the stern professor Stoletov. Rather quickly, I have managed to study this easy discipline within the lecture course. My reply to the questions formulated on the card was excellent. Suddenly, Stoletov asks: "What date is the longest day in Moscow?" I was completely taken aback by this question. My silence lasted several minutes. Stoletov was glassy staring at me and, at last, he said sluggishly: "Complete ignorance". He writes *unsatisfactory* in my record-book, and I am ruined because my marks for all other difficult exams were excellent. [...] It seemed that committing suicide was the best decision to suppress the feelings tearing my soul apart at that moment. However, this idea came to my mind only afterwards when I had already calmed down.

About science. The thought about committing suicide came to Steklov once more, when he was a second-year student in Kharkov. It was caused by his rather complicated love affairs. In his recollections [59], he writes in connection to this.

> Soon I came to the conclusion that any reasoning as to whether it is worth living or not is an inadmissible stupidity and moral cowardice. It is worth living for the sake of pursuit of knowledge and even my experience — rather small at that time — had already demonstrated that all other kinds of activity occupying people are deceptive and temporal. Research is the only kind of activity that occupies you forever and never deceives a person who wants and is ready to devote himself/herself to it. Soon, I immersed myself into studies once and for all. Moreover, the young professor Alexander Mikhailovich Lyapunov (my fellow countryman who, afterwards, became an outstanding mathematician) joined the faculty shortly after that. He was my teacher and only friend; his guidance of my first steps in science is unforgettable.

About students.[2] It is an amazing and lucky coincidence that the same year (1906) as Steklov got his professorship at the St. Petersburg University a group of very gifted students entered it to study mathematics. In the file of M. F. Petelin (he was one of them), this fact was noted by Steklov as follows.

> I should note that the class of 1910 is exceptional. In the class of 1911 and among the fourth-year students who are about to graduate there is no one equal in knowledge and abilities to Messrs. Tamarkin, Friedmann, Bulygin,

[02] In this section, all quotations are taken from the English version of [65].

> Petelin, Smirnov, Shohat and others. There was no such case during the fifteen years of teaching at the Kharkov University either. This favorable situation should be used for the benefit of the University.

This quotation as well as further ones concerning Steklov and his students show how attentive to them was he. Their future fate was very different; two of them (Bulygin and Petelin), unfortunately, died young.

A. A. Friedmann became famous for his discovery in general relativity; his solution of Einstein's equations was the first one that describes the expanding Universe (see [14] and [15]). However, he died aged 37, just seven months after his appointment as Director of the Main Geophysical Observatory in Leningrad and two months after his flight to the record altitude of 7,400 metres. V. I. Smirnov (a corresponding member of the Academy of Sciences of the USSR since 1932 and a full academician since 1943) is known for his results in complex analysis and mathematical physics. He is the author of *A Course in Higher Mathematics* (the first two of its five volumes were written in collaboration with Tamarkin but revised for later editions). From 1922 until his death in 1974, Smirnov's activity was associated with Leningrad University, where he founded the Research Institute for Mathematics and Mechanics in 1931 and afterwards headed several departments at the Faculty of Mathematics and Mechanics (Mat.–Mekh.). In the 1950s and 1960s, the Leningrad School of Mathematical Physics founded by Steklov flourished under the direction of Smirnov. His effort in restoring the Leningrad Mathematical Society in 1959 was also crucial. (The existence of its predecessor — the Physical–Mathematical Society — lasted from 1921 until 1930, when it disbanded due to political pressure; see [64].)

J. D. Tamarkin and J. A. Shohat emigrated to the USA in 1925 and 1923, respectively. They were active in research in various areas of analysis (the book [47] is their most cited work) and in supervising PhD students (G. Forsythe — one of Tamarkin's students — was afterwards a prolific PhD adviser himself). In 1927, Tamarkin was called to Brown University, whereas Shohat was at the University of Pennsylvania from 1930. Tamarkin was also involved in editing various journals; in particular, he was one of the editors of *Mathematical Reviews* when it started in 1940. As a member of the Organising Committee for the 1940 ICM, Tamarkin was very efficient. (Unfortunately, the congress was postponed because of World War II and took place after Tamarkin's death.) He was also an influential member of the AMS Council from 1931 and Vice President of the Society in 1942–1943.

In his recollections [59], Steklov also mentions A. S. Besikovitch who graduated in 1912 and was appointed to a professorship five years later at the newly opened Perm University (it was Steklov who recommended him). Besikovitch became famous for his contribution to the theory of almost periodic functions and for his results that form the cornerstones of geometric measure theory. He emigrated from the USSR in 1925 and, after staying one year in Copenhagen with Harald Bohr, moved to the UK. There, he became a university lecturer in Cambridge in 1927 and the Rouse Ball Chair of Mathematics in 1950. He had been elected F.R.S. in 1934 and received several academic awards.

In Steklov's diaries, the first mention of students is, chronologically, in the entry for 13 January 1908. What follows is a set of most important entries.

> 13 January 1908. At 4 o'clock Tamarkin and Friedmann (undergraduate students) turned up and brought the continuation of the lectures in integral calculus they had written. They took the ones I had corrected (i.e. looked through. No possibility of correcting them properly!). They said they would come to my lecture on the 16th. They asked me if it was possible to legalise the mathematical society without a supervisor. I told them to make some suggestions. Let us see!
>
> 20 February. Brought my collected works to the University and gave them to Tamarkin for the students' mathematical society. Three memoirs are missing.
>
> 21 October. Tamarkin and Friedmann came to see me this evening. They are going to organise a mathematics reading room. Asked me to be their supervisor. Declined, but they deserve help.
>
> 22 November. Tamarkin and Friedmann came to see me this evening [...] Kept asking me about their delvings into the theory of orthogonal functions. They are having an article published in Crelle's journal. Sharp fellows! They left at half past twelve, after supper.
>
> 18 April 1909. The students Tamarkin, Friedmann, Petelin came to see me this evening [...] I proposed to Tamarkin that he think about the asymptotic solution of differential equations (i.e. stability, in the sense of Poincaré and Lyapunov) or the problem of equilibrium of a rectangular plate. To Friedmann I suggested he find orthogonal substitutions, when fundamental functions are products of two (see my dissertation). I suggested Petelin read what Jacobi had to say about the principle of the last multiplier. I'll think it over again and will probably find some other topics too.
>
> 12 September. This evening Tamarkin, Friedmann and Petelin came to see me. They had worked on the assigned topics. Seem to have done something. Promised to submit their essays in a month. Tamarkin seems to be doing better than the others.

Steklov coauthored only two papers and one of them was a joint paper with Tamarkin. It was published in *Rend. Circ. Mat. Palermo* in 1911 and so was written when Tamarkin was still a student.

About administrative work in the Academy of Sciences. In Steklov's recollections [59], his comments on this topic are rather brief but they show that he clearly understood his role in the survival of the academy as the leading scientific institution.

> In 1919, I was unanimously elected to the post of Vice President of the Russian Academy of Sciences.[3] At the same time, the [Petrograd] University [...] insisted that I have to head it, but this burden was decidedly rejected by me. Indeed, the state-of-affairs existing at that time would not allow any human

[03] In 1925, it was renamed the Academy of Sciences of the USSR.

> to do both jobs properly. [...] It was absolutely clear to me that I could really do a lot for the benefit of the Academy. [...] On the other hand, I saw that the university was on the brink of collapse at that time.
>
> ... ...
>
> First, it must be said that the Academy is still one of a few institutions that were successfully vindicated from various destructive attacks. Moreover, its reputation was growing gradually in the eyes of ruling circles and now the Academy is recognised as the leading scientific institution. At last (in September 1923), I have achieved a success in the matter that I tried to accomplish for a long time, namely that the Academy must be considered on equal terms with Narkompros [the Ministry of Education]. [...] I can say with satisfaction, without boasting, that my contribution to achieving all these results favorable for the Academy is very considerable.

About Steklov's daily schedule. At the end of his recollections [59], one finds the following.

> In my opinion, it is exclusively due to a particular daily schedule that I manage to separate administration and research so that both of them flow parallel not interrupting each other. I adopted this schedule during my student days.
>
> My day is divided into two parts as follows. The time from 10 am to 5 or 6 pm I devote to administration at the Academy. Then I dine and about 7 pm go to bed. I sleep until 9:30 pm (sometimes until 10 pm). After awakening, I have a cup of tea and then, leaving apart all thoughts about administration and having nothing revulsive, calmly do my research.
>
> I work until 4 or 5 am in the morning (sometimes longer). [...] Three hours of sleep after dinner allow me to sleep from 4 or 5 am to 9:30 am, that is, 5 and sometimes 6 hours. This has been my daily schedule for more than 40 years and I find it expedient to a great extent.

Of course, it is difficult to stop quoting Steklov's recollections and diaries but enough is enough.

Steklov's unfinished monograph *Fundamental Problems of Mathematical Physics*

Mathematical achievements of the first half of the 20th century are described in the book [41]. Its first section entitled "Guidelines 1900–1950" is compiled by P. Dugac, B. Eckmann, J. Mawhin and J.-P. Pier with the assistance of an international team of almost six dozen prominent mathematicians (V. I. Arnold and S. S. Demidov represent the Moscow school). "Guidelines" is a year by year list of major results and their authors; the most important books published during the period from 1900 to 1950 are also presented in this 34-page list. It includes two items concerning mathematical physics published in 1923: *Lectures on Cauchy's problem in linear partial differential equations* by J. Hadamard and the two-volume book [57] by Steklov. Its second edition [58] appeared 60 years later with a vast number of comments and some necessary corrections made by V. P. Mikhailov and A. K. Gushchin (both from the Steklov Mathematical Institute, Moscow).

Here, my first aim is to explain why the treatise [57] is among the most valuable contributions to mathematical literature of the first half of the 20th century despite the fact that it was not finished by Steklov (see below). Secondly, further advances will be described in the area of applications of potential theory to boundary value problems for the Laplace equation, which is the topic of the second volume of [57].

Prior to that, it is worth mentioning several other contributions to "Guidelines" which came from Steklov's mathematical school. What follows are corrected excerpts from the list in [41] supplied with citations of the corresponding original papers:

1932 – A. S. Besikovitch, *Almost Periodic Functions.* Cambridge University Press.

1936 – S. G. Mikhlin, Symbol of a singular integral operator; [35], [36], [37] (a comprehensive updated presentation can be found in the monograph [38]).

1936 – S. L. Sobolev, First results on distributions; [49].

1937 – N. M. Krylov, N. N. Bogolyubov, Averaging method for nonlinear differential systems; [31].

1938 – S. L. Sobolev, Sobolev spaces; mollifiers; [50].

1950 – S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics.* (In Russian; English translation was published by the AMS in 1964.) This book presents results obtained in [49] and [50] in a comprehensive form.

Notice that the notion of mollifier proposed by Sobolev is a far-ranging generalisation of the *Steklov mean function*, which is the most simple averaging operator (see [1], § 74, for the definition and properties). It was introduced by Steklov in 1907 for studying the problem of expanding a given function into a series of eigenfunctions defined by a 2nd-order ordinary differential operator; see [54] and [55] for the announcement and full-length paper, respectively. Of course, "Guidelines" contains many other entries due to mathematicians from Russia, beginning with Steklov's teacher A. M. Lyapunov (1901 — Central Limit Theorem), and ending with several entries for 1950, one of which is M. G. Krein's "Parametric resonance in higher dimensional Hamiltonian systems". The overall best number of entries is 14, unsurprisingly, by A. N. Kolmogorov.

Let us turn back to the monograph [57]; it is based on lectures given to a small group of well prepared audiences in 1918–1920. This is why this book is written in Russian despite the fact that the underlying papers were written in French. Its 1st volume was finished in April 1919 and the 2nd volume was finished in November 1922. More material was presented in the lecture course than appeared in [57]; Steklov planned to publish the 3rd volume with his results concerning "fundamental" functions (that is, eigenfunctions of various spectral problems for the Laplacian) and some applications of these functions. He describes this objective on p. 257 of the 2nd volume and also mentions it in [59], p. 299. Unfortunately, his administrative duties as Vice-President of the Academy prevented him from realising this project. However, one gets an idea about the probable contents of the unpublished 3rd volume from the lengthy article [53]. In this paper, which appeared in 1904, Steklov developed his approach to "fundamental" functions (see [26], § 5, by A. Kneser for its brief outline). This approach uses two different kinds of Green's function and this allows one to apply the theory of integral equations worked out by I. Fredholm [13] and D. Hilbert [21] shortly before that.

In 1923 — the year when [57] was published — Russian scientists were still cut off from their colleagues in the West after the October Revolution. It is worth emphasising the great efforts of Steklov and his fellow academicians Abram Fedorovich Ioffe (he founded the Physical–Technical Institute in Petrograd in 1918), Alexei Nikolaevich Krylov (naval engineer and applied mathematician known for his work [30] winning a Gold Medal from the Royal Institution of Naval Architects) and Sergey Fedorovich Oldenburg (the Permanent Secretary of the Russian Academy of Sciences) directed towards restoring contact with colleagues abroad as well as to set up exchanges through scientific publications. Anyway, at that time no attempt was made to translate [57] into French, German or English. However, 11 years later, N. M. Günther (presumably he attended Steklov's lectures) gave an account of potential theory and its application to the Dirichlet and Neumann problems following the approach proposed by Steklov. First, Günther's book [19] was published in French, then its Russian revised and augmented edition appeared in 1953 and, finally, the English translation [20] of the latter was issued in 1967.

In his book, Steklov considers boundary value problems as mathematical models of physical phenomena and so two essential requirements must be fulfilled for a solution of any such problem: the existence and uniqueness theorems. This is the first important point of vol. 1. Notice that the notion of a well-posed problem was introduced simultaneously by Hadamard in his book mentioned above; he complemented these two requirements with the following one: a solution must depend continuously on the problem's data.

The second important point of vol. 1 is the systematic rigorous justification of the Fourier method for initial-boundary value problems for parabolic and hyperbolic equations with variable coefficients not depending on the time and depending on a single spatial variable. For this purpose the following stages must be accomplished.

- The existence of an infinite sequence of eigenvalues and eigenfunctions must be proved.
- It must be shown that the set of eigenfunctions is "rich enough" for expanding every "sufficiently smooth" function into a Fourier series.
- One has to prove that the obtained Fourier series gives a solution of the problem under consideration.

A detailed study of the Sturm–Liouville problem serves as the basis for the first two of these stages, and 6 of 11 chapters of vol. 1 are devoted to this problem. Steklov had written many papers on this topic (the first of them "On cooling of a heterogeneous bar" was published in Russian and dates back to 1896); the presentation of material in [57] follows his final article [56].

A great part of the contents of vol. 2 is based on Steklov's original contribution to the theory of boundary value problems for the Laplace equation: the two-part article [51], [52] published in 1902, the second of which is the most cited of Steklov's work. Confusingly, his initials are given incorrectly in almost all its citations. Indeed, R. Weinstock mistook the abbreviation "M. W." ("M." stands for "Monsieur" in French) for Steklov's initials and this was afterwards reproduced elsewhere. Nevertheless, we must be grateful to Weinstock for introducing the term "the Steklov problem" in his paper [69] published in 1954, in which he initiated studies of the following problem:

$$\nabla^2 u = 0 \ \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda \varphi u \text{ on } \ \partial D. \tag{1}$$

In fact, Steklov proposed this problem in his talk at a session of the Kharkov Mathematical Society in December 1895; nowadays, it is mainly referred to as the Steklov problem but, sometimes, it is also called the *Stekloff problem* as in [69].

In problem (1), D is, generally speaking, a bounded Lipschitz domain in $\mathbb{R}^m$, n is the exterior unit normal existing almost everywhere on ∂D, and λ denotes the spectral parameter. This problem is similar to the spectral problem for the Neumann Laplacian in the following sense. The latter problem describes the vibration of a homogeneous free membrane, while the Steklov problem models the vibration of a free membrane with all its inhomogeneous mass $\varphi \geq 0$, $\varphi \not\equiv 0$ concentrated along the boundary (see [3], p. 95).

In [69], an isoperimetric inequality is proved for the smallest positive eigenvalue of (1) under the following assumptions: $m = 2$, whereas $\varphi \equiv 1$ on ∂D which is an analytic curve. Further progress achieved about inequalities for eigenvalues of problem (1) and other related problems can be found in the book [3] by C. Bandle, in § 8 of the survey article [2] by M. S. Ashbaugh and R. D. Benguria, and also in the recent papers [4], [17] and [18] by I. Polterovich and his coauthors.

In the opinion of Steklov's contemporaries (see [26], § 6, and two papers by Günther in [16]), which is shared by the compilers of "Guidelines", his results presented in vol. 2 are of paramount interest. They deal with the Dirichlet and Neumann problems in interior and exterior domains separated by a closed surface in $\mathbb{R}^3$. Steklov was the first who proved the existence of solutions to these problems by means of potential theory in the case of an *arbitrary* (that is, without any shape restriction) $C^{1,\alpha}$-surface, $\alpha \in (0, 1]$. (These surfaces are also referred to as Lyapunov's because they were introduced by him in [33].) In order to prove the existence of solutions Steklov used iterative procedures aimed at finding the densities of the double and single layer potentials which solve the Dirichlet and Neumann problem, respectively. Thus, a definitive solution had been given to a longstanding question concerning these problems. However, unlike many other definitive solutions, Steklov's did not kill the field and further developments are outlined in the next section.

Potential-theoretic approach to boundary value problems for the Laplace equation

The method which is standard in textbooks nowadays is as follows. Potential theory is applied for reducing the Dirichlet and Neumann problems to integral equations, which are then investigated with the help of Fredholm's theorems. It seems that it was O. D. Kellogg who first realised this approach in detail in his comprehensive monograph [23]. However, his assumption, that a surface dividing $\mathbb{R}^3$ into two domains belongs to the class C^2, is superfluous. V. I. Smirnov applied the same approach in the case of Lyapunov's surfaces in his classical textbook [48] (its 1st edition was published in 1941). This assumption is sufficient to guarantee that the kernels of arising integral operators have a weak (polar) singularity.

As early as 1916, T. Carleman [9] initiated studies of boundary value problems for the Laplace equation in domains with non-smooth boundaries. In particular, he developed a potential-theoretic approach to the Dirichlet and Neumann problems in the case when a surface dividing $\mathbb{R}^3$ into two domains consists of several pieces each belonging to the C^2 class and overlapping pairwise along edges which are C^2-curves. Moreover, half-planes tangent to two different pieces must not coincide at every edge-point. The method used by Carleman in three dimensions is a straightforward generalisation of his technique applied for two-dimensional domains with a finite number of corner points on the boundary (see also [32], § 2.1.3, where this technique is outlined).

In 1919, J. Radon [42] made the next step in developing the potential-theoretic approach for irregular domains in two dimensions. He considered the corresponding integral operators for contours having "bounded rotation" without cusps (see also the survey paper [34], ch. 4, § 1, for the exact definition). It took more than 40 years to generalise Radon's result to boundary value problems in irregular higher-dimensional domains. As often happens, this was accomplished simultaneously in two different places: in Leningrad and in Prague (see [7], published by mathematicians from Steklov's school, and [27], respectively). One can find further details in [6] and [28] (see also [34], ch. 4, § 2). In particular, it was shown that the square of C. Neumann's operator (the latter is also referred to as the direct value of the double layer potential) is a contraction operator on the boundary of a convex domain (see the paper [29] by J. Král and I. Netuka, and also Král's lecture notes [28], § 3). Moreover, converging iterative procedures were developed in this case for the integral equations of interior and exterior Dirichlet and Neumann problems. These procedures involve Neumann's operator (and its dual, respectively) and are similar to those proposed by Steklov in the case when an arbitrary Lyapunov surface divides $\mathbb{R}^3$ into two domains.

During the last quarter of the 20th century, the potential-theoretic approach to the Dirichlet and Neumann problems for the Laplace equation was devised for C^1 and Lipschitz domains. The beginning to this development was laid by A. P. Calderon's note [8], in which boundedness of the Cauchy singular integral was proved in L^p over a Lipschitz curve provided the Lipschitz constant is sufficiently small; this restriction was later removed for L^2 (see [10]). These results allowed the investigation of solubility of boundary integral equations for problems with L^p boundary data (see [12] and [67] for the case of

C^1 and Lipschitz domains, respectively). A brief review of these results is given in the survey article [34], ch. 4, § 3, whereas the book [24] by C. E. Kenig contains their systematic exposition, some generalisations and an extensive list of references. Besides this, a simple treatment of boundary integral operators on Lipschitz domains was proposed by M. Costabel [11].

Furthermore, B.-W. Schulze and G. Wildenhain [45] presented results concerning potential theory for higher order elliptic equations and covered the usual topics as in the classical case; general boundary value problems, strongly elliptic systems and problems in Beppo Levi spaces are also considered.

In conclusion of this section, one more development of Steklov's approach to iterative solutions of boundary integral equations should be mentioned. In the case of the exterior Dirichlet problem for the Laplace equation, he used iterations which give the problem's solution although the corresponding homogeneous integral equation has a non-trivial solution. Besides, if one applies potentials involving the standard fundamental solution of the Helmholtz equation for solving exterior boundary value problems then the corresponding homogeneous integral equations usually have non-trivial solutions for several values of the frequency problem's parameter. (The values for which the method fails are referred to as *irregular* frequencies.) Nevertheless, it is possible to modify a boundary integral equation so that a properly transformed iteration method gives its solution for all frequencies. It was shown by R. E. Kleinman and G. F. Roach [25] that one has to use a modified Green's function and to adapt an iteration procedure for this purpose. The modified Green's function is equal to the sum of the standard fundamental solution and a series of some given solutions of the Helmholtz equation with properly chosen coefficients. This technique was introduced in the 1970s and further developed in the 1980s (see references cited in [25]); it allows one to obtain integral equations without irregular frequencies at the expense of losing self-adjointness of the integral operator. Furthermore, for an integral equation with a modified Green's function there exists the iterative solution converging to the exact one as a geometric progression.

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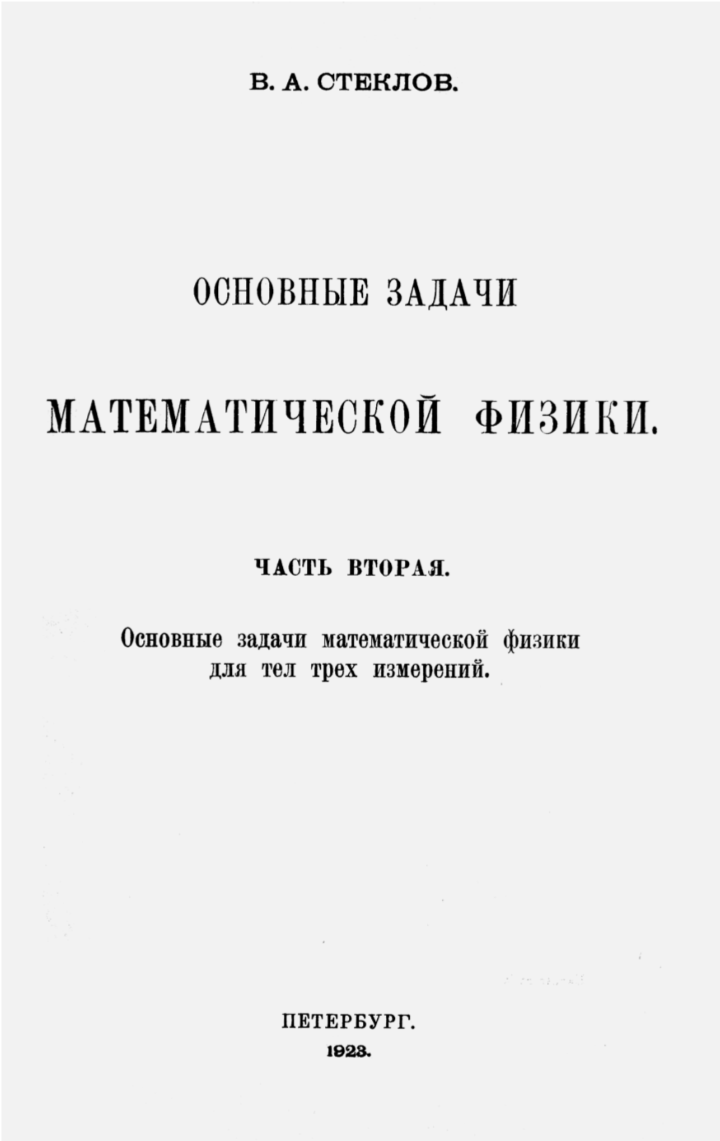

В. А. СТЕКЛОВ.

ОСНОВНЫЕ ЗАДАЧИ

МАТЕМАТИЧЕСКОЙ ФИЗИКИ.

ЧАСТЬ ВТОРАЯ.

Основные задачи математической физики
для тел трех измерений.

ПЕТЕРБУРГ.
1923.

The title page of the Steklov's monograph [57], Vol.2.

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IST Austria and Mathematical Physics

by László Erdős and Robert Seiringer (IST Austria)

Vienna is a familiar city to many IAMP members through the Erwin Schrödinger Institute (ESI) dedicated to mathematical physics in the broadest sense. In this article we present a new Austrian scientific institution, the Institute of Science and Technology Austria (IST Austria), where we took up professor positions last year. While the scope and purpose of IST Austria is very different from that of ESI, we hope that our new institute will take an equally prominent place in mathematical physics. We explain the unique structure of IST Austria and various opportunities it may bring to our community.

The mission of IST Austria

The Institute of Science and Technology Austria (IST Austria) is a multidisciplinary research institution dedicated to cutting-edge basic research in the natural, mathematical, and computer sciences, both theoretical and experimental. It is not a university in the traditional sense; its focus is on curiosity-driven research and training of graduate students. The long term financial commitment from the Austrian federal government and from the state of Lower Austria, as well as the governance and management structures of IST Austria, guarantee the Institute's freedom from political and commercial influences.

We believe IST Austria is unique in the world; among other institutions world-wide its structure is probably the closest to the Weizmann Institute. In fact, Professor Haim Harari, former president of the Weizmann Institute, has played a pivotal role in setting up the concept of IST Austria. The vision was to create a special environment where researchers from different disciplines have much more interaction than at a traditional institute with separate faculties, departments and schools.

To foster a creative and interdisciplinary scientific atmosphere, all hierarchical and separating organizational structures, such as departments, are avoided at IST Austria. The scientists are organized into independent research groups, each headed by a professor or a tenure-track assistant professor.

Jointly designed courses, interdisciplinary seminars, an institute-wide hiring process and, last but not least, joint social activities open to all members, including our cafe, retreats, table-tennis tournaments and the like, bring together mathematicians, physicists, computer scientists and biologists. Interdisciplinary research is not a requirement, as scientific excellence is the only criterion for evaluating our performance, but the open environment strongly fosters interactions. It is less important whether these interactions result in joint publications, grant-proposals, co-supervised students, or only in open discussions on scientific issues crossing the boundaries of traditional disciplines. More important is that a culture of openness is established and passed on to our students.

We do not have undergraduate students, but as a PhD granting institution, IST Austria educates doctoral students from around the world. The official language on the whole campus is exclusively English to ensure that international students and faculty members can equally participate in all scientific and social activities.

Facts and figures

IST Austria was established jointly by the federal government of Austria and the provincial government of Lower Austria, and was inaugurated in 2009. The Institute is located at a new campus in the Vienna Woods near the city of Klosterneuburg, 18 km from the center of Vienna. The Vienna subway system is accessible by a special shuttle service as well as with regular public buses in 25 minutes (for visitors of the ESI, we are less than 45 minutes away!)

In the fall of 2010, the first laboratory building, the Bertalanffy Foundation Building, was opened and the first experimental research groups started their work at the Institute. The second laboratory building, Lab Building East, was completed at the end of 2012. Further buildings are under construction or in the planning phase.

For the period from 2007 until 2026, the federal government of Austria provides up to 1 280 million Euro in operational funds. Two thirds of this budget are guaranteed, while the remaining third is conditioned on performance-related criteria such as the raising of third-party funds. The state of Lower Austria contributes the budget for construction and campus maintenance, in a total amount of 510 million Euro from 2007 until 2026. By the end of 2013, IST Austria has obtained commitments for more than 36 million Euro in research grants, the vast majority of which originates from sources outside of Austria, among them 14 ERC grants for members of the faculty. IST Austria is also active in fundraising and, by the end of 2013, has received more than 17 million Euro in donations.

At the end of 2013, there were 28 independent research groups, each led by a professor or an assistant professor. Independence is especially important and attractive for younger scientists in experimental areas; IST Austria gives them the opportunity to build their own labs. The development plan of IST Austria allows for a growth to 90–100 research groups, altogether 1000 scientists, by 2026, i.e., we plan to hire 5–6 new faculty members every year in the next decade.

Faculty and hiring

To keep the hierarchy minimal, there are only two levels of professors. Younger scientists are hired as tenure-track assistant professors. Their promotion to the level of professor with a permanent contract is similar to the tenure procedure at leading US institutions. Research excellence and promise are the exclusive hiring criteria for all scientists at IST Austria — from doctoral students to professors.

The Institute's choice of scientific topics is based solely on the availability of outstanding individuals: a direction of research is pursued only if IST Austria can compete with the best in the world. The Institute does not allocate a predetermined number of positions to different disciplines and every faculty member participates in every hiring discussion in a very open style. However, there is a strong belief in and support for diversity among faculty members; areas that are less represented receive more attention in hiring. Currently half of our faculty works in life sciences, followed by computer science as the second most represented area. Mathematics and physics still lag behind, but concentrated efforts in the last two years have improved the balance.

IST Austria is headed by the President, who is appointed by the Board of Trustees and advised by the Scientific Board. The first President of the Institute is Thomas A. Henzinger, a computer scientist and former professor of the University of California at Berkeley and the EPFL in Lausanne, Switzerland, who started his second term in 2013.

Mathematical physics at IST Austria

Interdisciplinary by definition, mathematical physics was a natural addition to IST Austria. We were hired simultaneously with the expectation to help build up applied mathematics and theoretical physics. We share a common platform in mathematical physics, with Robert having more background in physics while László has more anchor in mathematics. Both of us have independent research groups that each will consist of 2-3 postdocs and graduate students. A very talented young theoretical physicist, Misha Lemeshko, will join IST Austria in the fall of 2014 as assistant professor. The Institute is currently very actively searching for experimental physicists. Mathematics was already represented on campus but mainly by discrete mathematicians. For a healthy balance, we hope to hire colleagues in other main directions such as differential equations, probability theory, geometry or algebra.

Opportunities

IST Austria is constantly seeking for applications for full professor positions from experienced scientists with international reputation. Every fall we also have a call for applications on the tenure-track level; a typical applicant is 3-8 years after Ph.D. and has already an outstanding record of independent research. Since hiring is done on an institute-wide level, attracting many excellent candidates in mathematics and physics is a key to strengthening the presence of our disciplines at IST Austria.

For scientists with fresh Ph.D., IST Austria has a centrally financed postdoctoral program, the ISTFellowship. Applications are solicited twice a year (deadlines: March 15, September 15) and about 6–8 two-year research positions are available each time. The selection is done by a single committee hence candidates from very different disciplines compete. Phan Thanh Nam is currently an ISTFellow in Robert's group, Christian Sadel and Dirk Zeindler will join László's group in the spring 2014 as new ISTFellows in mathematical physics.

Our graduate school admits students once a year; the application deadline is January 15 for a start in the fall semester. Currently we admit about 30 students per year, but the size of the graduate class will increase as IST Austria expands. Students are provided with a generous fellowship for the whole duration of their studies assuming satisfactory progress. In the first year the students prepare for their qualifying exam similarly to a typical US graduate school. In addition to standard course requirements, our students participate in three "rotation" programs. A rotation is a shorter (three month) project with a research group. This system helps students identify their interest and find an advisor for their thesis. After the qualifying exams, the students affiliate with their chosen research group. Since IST Austria does not have undergraduate education, the duties of our graduate students as teaching assistants are minimal.

We have recently established a long-term visitor program with partial financing, designed for experienced scientists spending a 3–12 month period at IST Austria; this opportunity is primarily aimed at people looking for sabbatical opportunities.

Finally, we have an internship program (called ISTernship) offered to talented and motivated undergraduate students who wish to spend 8–12 weeks during the summer at IST Austria. This program is especially recommended to students who plan to apply to our graduate school.

In addition to the central programs, every research group has its own research budget. On the discretion of the group leader, this budget can also be used to hire postdocs or invite visitors.

A more detailed description of these opportunities, as well as up-to-date information on IST Austria, can be found at our website at http://www.ist.ac.at, where it is also possible to sign up for the Institute's quarterly newsletter. We ask IAMP members to spread the information on these opportunities to qualified candidates.

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. Mohammed Al Amr, Department of Mathematics, University of Mosul, Iraq
2. Dr. Atsuhide Ishida, Economics, Otemon Gakuin University, Osaka, Japan
3. Dr. Jishad Kumar, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
4. Prof. Andrew S. Lang, Department of Mathematics, Oral Roberts University, Tulsa, OK, USA
5. Dr. Zouhair Mouayn, Department of Mathematics, University Sultan Moulay Slimane, Béni Mellal, Morocco.
6. Dr. Simona Rota Nodari, Laboratoire Paul Painlevé, Université Lille 1, France
7. Prof. Enrique Valderrama Araya, Department of Mathematics, Oral Roberts University, Tulsa, OK, USA

Recent conference announcements

Frontier Probability Days

May 18-20, 2014, University of Arizona, Tucson, AZ, USA.

organized by
Tom Alberts, Davar Khoshnevisan, Firas Rassoul-Agha, Sunder Sethuraman, Edward C Waymire

Web page http://math.arizona.edu/~fpd.

Nonequilibrium Problems in Physics and Mathematics

June 1-6, 2014, Centro Stefano Franscini, Monte Verita, Ascona, Switzerland

organized by
Jean-Pierre Eckmann and Antti Kupiainen

Web page http://theory.physics.unige.ch/NPPM/

Spectral Days 2014

June 9-13, 2014, CIRM Marseille, France

Web page http://www.cirm.univ-mrs.fr/index.html/spip.php?rubrique2&EX=info_rencontre&annee=2014&id_renc=1041&lang=fr

organized by
Simone Warzel, Jean-Marie Barbaroux, François Germinet, Alain Joye.

This conference is partially funded by the IAMP.

Solid Math 2014

June 16-18, 2014, SISSA (Trieste)

Web page https://sites.google.com/site/solidmath2014/home

organized by
Gianfausto Dell'Antonio, Alessandro Giuliani, Domenico Monaco, Gianluca Panati

This conference is partially funded by the IAMP.

NSF/CBMS Regional Conference on Quantum Spin Systems

June 16-20, 2014, University of Alabama at Birmingham, USA.

The distinguished lecturer will be Bruno Nachtergaele (UC Davis), who will give 10 lectures accessible to newcomers to the field of quantum spin systems, but leading to advanced topics and open problems. The program also includes additional lectures by other experts in the field.

organized by Shannon Starr, Paul Jung, Gunter Stolz

Web page http://www.uab.edu/cas/mathematics/events/nsf-cbms-conference-2014

Mathematics Meets Physics

June 24-27, 2014, Helsinki, Finland

A four day conference exploring the frontiers of mathematical physics on the occasion of Antti Kupiainen's 60th birthday.

The aim of the conference is to foster the exchange of recent breakthroughs, new ideas and advances in methodology by bringing together world-leading experts, ranging the full spectrum from pure mathematics to physics.

Web page http://wiki.helsinki.fi/display/mathphys/mathphys2014

Quantum Roundabout

Student conference on the mathematical foundations of quantum physics

June 29-July 3, 3014, University of Nottingham, UK

Web page http://quantumroundabout.weebly.com

organized by Gerardo Adesso, Thomas Bromley, Ioannis Kogias

This conference is partially funded by the IAMP.

XXX$^{\text{th}}$ International Colloquium on Group Theoretical Methods in Physics

July 14-19, 2014, Ghent University, Ghent, Belgium

Web page http://www.group30.ugent.be

organized by
Joris van der Jeugt, Jean-Pierre Antoine, Francoise Bastin, Pierre Bieliavsky, Fred Brackx, Stefaan Caenepeel, Frans Cantrijn, Hennie De Schepper, Simone Gutt, Marc Henneaux, Erik Koelink, Piet Van Isacker, Pierre Van Moerbeke

This conference is partially funded by the IAMP.

Summer School on Mathematical Physics

July 21-26, 2014, Universität Heidelberg, Germany

Web page http://www.thphys.uni-heidelberg.de/summerschool2014

organized by
Christoph Kopper and Manfred Salmhofer

This summer school is partially funded by the IAMP.

99 years of General Relativity: The ESI-EMS-IAMP Summer school on Mathematical Relativity

July 28-August 1, 2014, The Erwin Schrödinger Institute for Mathematical Physics, University of Vienna, Austria

Web page http://homepage.univie.ac.at/piotr.chrusciel/SummerSchool2014/

organized by
Robert Beig and Piotr T. Chruściel.

This summer school is partially funded by the IAMP.

Trimester program on Non-commutative Geometry and its Applications

September-December, 2014, Hausdorff-Institut für Mathematik, Bonn, Germany

organized by
Alan L. Carey, Victor Gayral, Matthias Lesch, Walter van Suijlekom, Raimar Wulkenhaar

Web page http://www.him.uni-bonn.de/programs/future-programs/future-trimester-programs/non-commutative-geometry-2014/description

Selected Problems in Mathematical Physics

September 1-5, 2014, La Spezia, Italy

organized by
Riccardo Adami, Michele Correggi, Rodolfo Figari, Alessandro Giuliani

Web page http://sp2014.tqms.it/index.html

This conference is partially funded by the IAMP.

MPAG: new editorial board

Starting from February 15, 2014, the Springer journal MPAG: Mathematical Physics, Analysis and Geometry has a new editorial board, formed by:

Jeremie Bouttier (CEA Saclay), Damien Calaque (Montpellier 2), Patrik Ferrari (Bonn), Rupert Frank (Caltech), Cristian Giardina' (Modena), Alessandro Giuliani (Roma Tre), Antonio Ponno (Padova), Benjamin Schlein (Zurich), Walter D. van Suijlekom (Nijmegen)

Submission of papers is invited in several different areas of mathematical physics, including quantum field theory, many body theory, spectral theory, combinatorics, classical mechanics, equilibrium and non-equilibrium statistical mechanics, integrable systems.

Open positions

Professorship in Mathematical Physics at ETH Zurich

The Department of Mathematics at ETH Zurich invites applications for a full professor position in Mathematics with a focus in Mathematical Physics. We are seeking candidates with an outstanding research record and a proven ability to direct research work of high quality. Willingness to participate in collaborative work both within and outside the school is expected. Furthermore, the new professor will be responsible, together with other members of Department, for teaching undergraduate (German or English) and graduate courses (English) for students of mathematics, natural sciences and engineering. Please apply online at http://www.facultyaffairs.ethz.ch. Applications should include a curriculum vitae, a list of publications, and a statement of your future research and teaching interests. The letter of application should be addressed to the President of ETH Zurich, Prof. Dr. Ralph Eichler.

The closing date for applications is 30 April 2014.

Postdoctoral Position in Mathematical Physics in Santiago, Chile.

The research group in Mathematical Physics of the Pontificia Universidad Catlica de Chile (PUC) invites applications for a one year (renewable to a second year) postdoctoral fellowship beginning August 1, 2014. Applicants should have a recent Ph.D. in mathematics or physics and should work in the group's research area (quantum mechanics, spectral analysis, scattering theory, functional analysis). The postdoctoral fellow is expected to interact with group members and should therefore be able to communicate in either English, Spanish or French. The fellowship involves no teaching. The annual stipend of Ch$ 18 000 000 is tax-free but compulsory Chilean medical insurance is required. The PUC is a leading Chilean research university with strong doctoral programs in mathematics and physics. A group headed by Rafael Benguria has been funded by the Chilean Government?s Iniciativa Cientifica Milenio for a three year period beginning in January 2014, to conduct research in Mathematical Physics. Other members of the group are G. Raikov, R. Tiedra (Math. at PUC), M. Mantoiu (Math. at U. de Chile) and M. C. Depassier, and E. Stockmeyer (Physics, PUC). Applicants should arrange that a curriculum vitae, a research statement and two letters evaluating the applicant's research be mailed to Rafael Benguria (rbenguri@puc.cl). In addition, applicants are encouraged to contact relevant group members directly.

For full consideration, complete application materials must arrive by May 31, 2014. Review of applications will begin on June 1.

Postdoctoral Position in Mathematical Physics at the University of Jena

A postdoctoral position in Mathematical Physics will be available on October 1st, 2014 at the University of Jena. The appointment is for 2 years with the possibility of extension. Please send applications including (i) CV (ii) list of publications (ii) statement of research interests (iv) a letter of recommendation to david.hasler@uni-jena.de.

For more information see the official announcement http://www.uni-jena.de/Universit%C3%A4t/Stellenmarkt/Wissenschaftliche+Mitarbeiter/47_2014+_+wiss_+Mitarbeiter_in +am+LS+Mathematische+Physik.html

The deadline for applications is June 15, 2014.

More job announcements are on the job announcement page of the IAMP

`http://www.iamp.org/page.php?page=page_positions`

which gets updated whenever new announcements come in.

Manfred Salmhofer (IAMP Secretary)

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