

International Association of Mathematical Physics



# News Bulletin

January 2015



# International Association of Mathematical Physics News Bulletin, January 2015

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*Cover picture:* All our best wishes for a happy and peaceful year 2015

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## New IAMP Executive Committee

by ROBERT SEIRINGER (President of the IAMP)



Following the election held in the past fall, a new Executive Committee (EC) will lead the IAMP in the coming three years. Let me take this opportunity as the newly elected president to thank the previous EC and its officers for the excellent work they have done for the IAMP. I think it is fair to say that the IAMP's state has improved significantly in recent years. As announced in the various IAMP bulletins, many new members have joined the association, and the total number of paying members is now above 700. The resulting improvement of the association's finances has allowed IAMP to increase its support for conferences and workshops, and hence directly benefits the mathematical physics community.

I am also grateful to the members of the newly appointed EC for their willingness to participate in this effort. Our goal is to further increase our membership, our visibility and, in particular, the breadth of scientific areas covered. The need for a broader membership base is clearly reflected in the current list of committee members. While they are all excellent scientists and are all highly respected in the community, one cannot help but notice that there is a clear bias concerning the type of science they do. This is a reflection of the distribution of our members across the various disciplines, and shows that there is a need for attracting further members working in currently underrepresented fields. On the positive side, I would like to note that the average age of the EC members is now significantly lower than in previous years, which is certainly a positive sign for the future.

In the past the scientific committees in charge of the ICMP's program have done an excellent job of trying to have a broad range of fields in mathematical physics represented at the congress. This makes the ICMP a particularly interesting event to attend, in my opinion. The fact that many of the ICMP plenary and session speakers are not members of IAMP shows that there is a clear potential for further increasing the number of members. I encourage you all to help in this endeavor, by spreading the word and encouraging colleagues to join. The larger, and scientifically broader, the list of members, the stronger the community will be.

Let me close by expressing my sincere hope that many of you will attend the upcoming ICMP in Santiago de Chile this coming July. You can check the website (<http://www.icmp2015.cl>) for updates on the scientific program and the many exciting events taking place. I would also like to encourage you to seriously consider making a bid for hosting the next ICMP

Robert Seiringer

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in 2018. While this is certainly a demanding and time-consuming task, the effort will be highly appreciated by the community.

See you in Chile in July!

## A Fields Medal for Martin Hairer

by JEAN-PIERRE ECKMANN (Geneva) and JEREMY QUASTEL (Toronto)

Before starting to work on KPZ and regularity structures, Martin's papers covered a wide range of important fields: Stochastic PDE's in infinite dimensions [3, 4, 7], uniqueness of the invariant measure for 2D Navier Stokes [5], transport properties of very singular heat conduction problems [6]. And it is perhaps of interest to point to another field where Martin excels:

*The Swiss army knife of sound editing!*

(He was raised in Switzerland.) Indeed, Martin's programming skills are known to many Disk-Jockeys through his award winning program "Amadeus" (for the Mac). And of course, these skills also show through in much of his mathematical work, and through wavelets the two are not entirely unrelated. One should note that his work on Amadeus started well before he entered university.

But, of course, this report is about *noise* not *music*; we want to explain what he got the Fields medal for, which is his spectacular recent work on non-linear stochastic partial differential equations. A good representative example is the one dimensional Kardar-Parisi-Zhang (KPZ) equation, which serves as a canonical physical model for the motion of an interface between a stable state expanding into a metastable state in two dimensional systems (one space, one time).

If both states are stable, the interface, which we assume is given by a height function  $h(t, x)$ , is driven by two main effects, relaxation, modeled by a heat flow, and a driving noise  $\xi(t, x)$  which is idealized to be uncorrelated in space and time, *i.e.*, Gaussian white noise. The equation of motion is then the Langevin equation,

$$\partial_t h = \partial_x^2 h + \xi. \quad (1)$$

This is the *Edwards-Anderson model*, and although the driving noise is very singular, it is rather easy to make sense of it. If  $P$  is the heat kernel, the solution to (1) could really be nothing but

$$h = P \star \mathbf{1}_{t>0} \xi + P h_0. \quad (2)$$

Given the initial condition, this is just a Gaussian variable with mean zero and covariance  $\langle h(t, x); h(t', x') \rangle = \int_0^{\min t, t'} \int_{-\infty}^{\infty} P(t-s, x-y) P(t'-s, x'-y) dy ds$ , and from this it is not hard to see that for fixed  $t > 0$  it is locally Brownian in space. Furthermore, the result is stable in the sense that if we took smooth approximations  $\xi_\varepsilon(t, x)$  to our space-time white noise, perhaps by convolving with a kernel of width  $\varepsilon$ , then the resulting solutions  $h_\varepsilon(t, x)$  to the approximating equation,  $\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \xi_\varepsilon$ , would converge to our  $h(t, x)$ .

This remains true for any reasonable regularization. Equation (1) is self-similar under the  $-1 : 2 : 4$  rescaling

$$h(t, x) \mapsto N^{-1}h(N^2x, N^4t), \quad (3)$$

which can be interpreted as saying that an initially nice interface will develop fluctuations of size  $t^{1/4}$  by time  $t$ .

When, instead, one of the states is metastable, one modifies (1) by adding a nonlinear drift, which reasonably should only depend on  $\partial_x h$  instead of the frame  $h$ :

$$\partial_t h = F(\partial_x h) + \partial_x^2 h + \xi. \quad (4)$$

Expanding

$$F(\partial_x h) = F(0) + F'(0)\partial_x h + \frac{1}{2}F''(0)(\partial_x h)^2 + \dots, \quad (5)$$

the first two terms can be removed by simple changes of coordinates, and this leads to the celebrated KPZ equation

$$\partial_t h = \lambda(\partial_x h)^2 + \partial_x^2 h + \xi. \quad (6)$$

It is here that the puzzling difficulty appears which was solved by Hairer. For each  $t > 0$ ,  $h(t, x)$  is locally Brownian in the  $x$  variable, but we are asked to differentiate it (locally white noise) and then square the result! So it is a real challenge to make sense of (6), as well as of (5).

It would not matter so much if (6), as well as its cousins, such as the continuous parabolic Anderson model in 2 and 3 dimensions

$$\partial_t u = \Delta u + u\eta, \quad (7)$$

where  $\eta$  is white noise in space only, were not so important. Another example is the dynamic  $\Phi^4$  model in 3 dimensions,

$$\partial_t \Phi = \Delta \Phi + \Phi^3 + \xi. \quad (8)$$

Each of these equations is a model for non-Gaussian fluctuations for its own, large, universality class (for (8), think of the fluctuation field of the Glauber dynamics for the critical Ising model). However, all these problems sat well outside the existing theory of stochastic partial differential equations.

The one dimensional KPZ has, in addition, deep connections to integrable systems, which have led to a series of remarkable exact solutions [2]. Unlike (1), it is not scale invariant. Instead, it converges to (1) for large  $N$  in the  $-1 : 2 : 4$  scaling (3) but also to a non-trivial *KPZ fixed point* in the  $-1 : 2 : 3$  scaling

$$h(t, x) \mapsto N^{-1}h(N^2x, N^3t). \quad (9)$$

Now the interface at time  $t$  has locally Brownian fluctuations of size  $t^{1/3}$ . About the fixed point rather little is known: What we have, from exactly solvable models in the universality class, are several self-similar solutions, the so-called *Airy processes*. The

KPZ equation also arises as universal *weakly asymmetric* or *intermediate disorder limits* of models which have a non-linearity or noise of size  $N^{-1}$  in the  $-1 : 2 : 4$  large  $N$  scaling limit (3). This scaling steers them towards the EW fixed point (1), but at the last second they bifurcate to follow the KPZ equation (6). Proofs are available in special cases, by exponentiating, and showing convergence towards the solution of the well-posed multiplicative stochastic heat equation,

$$\partial_t Z = \partial_x^2 Z + \xi Z. \quad (10)$$

The *Hopf-Cole solution*,

$$h(t, x) = \log Z(t, x) \quad (11)$$

is the true solution of (6). Unfortunately, the trick works only when the exponentiated height function happens to satisfy a nice discrete version of (10).

Now suppose we want to make some sense of the passage from (5) to (6) as a weakly asymmetric limit. We start with

$$\partial_t h = N^{-1} F(\partial_x h) + \nu \partial_x^2 h + \zeta. \quad (12)$$

where  $F$  is an arbitrary even function, and  $\zeta$  is a smooth Gaussian random field in space and time, with finite range correlations. After the  $-1 : 4 : 2$  scaling (3), it should converge to the KPZ equation (6). But it is clear that in this case the approximation to (10) will be horrible to deal with, and this emphasizes why one really needs to give intrinsic meaning to the KPZ equation (6) itself. For other non-linear stochastic equations, one does not have analogous tricks.

We start with the integral form of the equation, which in the case of KPZ (6) would be

$$h = P \star \mathbf{1}_{t>0}((\partial_x h)^2 + \xi) + Ph_0. \quad (13)$$

Of course, the problem is to find a consistent way to make sense of a non-linear function of a distribution. For technical reasons, one works on a finite interval with periodic boundary conditions. Now of course, there is a classical solution map  $(\xi, h_0) \mapsto h(t, x)$  which, unfortunately, is only defined for *nice* inputs  $\xi$  and  $h_0$ . We would like to take the first to be a smooth approximation  $\xi_\varepsilon$  to our *very not nice* noise  $\xi$ , and somehow take a limit of the resulting solution  $h_\varepsilon(t, x)$ . This turns out to be too much to ask. However, it is true that the solution  $h_\varepsilon(t, x)$  to an appropriate renormalized equation does converge. In the KPZ case, it is achieved simply by subtracting an appropriate constant  $C_\varepsilon$ ;

$$h_\varepsilon = P \star \mathbf{1}_{t>0}((\partial_x h_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon) + Ph_0. \quad (14)$$

Hairer's main result is that there is a (random)  $h(t, x)$  such that  $h_\varepsilon \rightarrow h$ , in probability, locally uniformly as continuous functions, that it coincides with the Hopf-Cole solution (11), and that it would be the same if we used any other reasonable regularization procedure. The result is analogous for other stochastic partial differential equations such as (7) and (8), except that one may have to act on the equation by a renormalization with a finite number of parameters, the *renormalization group*,  $\mathfrak{R}$ . The key condition

is *local sub-criticality*, which roughly means that when you zoom into the equation, the non-linearities go away.

His basic tool is the notion of *regularity structures*. This is an abstract vector space with enough information to provide a local description of the solution. It should contain abstract polynomials in the time and space variables, as well as a symbol  $\Xi$  representing the input noise, rules for multiplication, and abstract versions  $\mathcal{P}$  of convolution with the heat kernel and  $\mathcal{D}$  of differentiation, and transfer rules for moving from a description at one point, to another. The symbols all come with homogeneities, which in the case of monomials is just their degree. But in parabolic space-time, and because the homogeneity of  $\Xi$  is something just below  $-3/2$ , there have to be terms of negative homogeneity. The abstract fixed point problem is

$$H = \mathcal{P}((\mathcal{D}H)^2 + \Xi) + Ph_0. \quad (15)$$

But this is too abstract: A *model* on the regularity structure is a map to distributions which turns these homogeneities into concrete analytic behavior. And it is *admissible* if it plays nicely with the multiplication rules as well as the operators  $\mathcal{P}$  and  $\mathcal{D}$ . It is in the latter that the highly non-trivial structure arises.

Once one has an admissible model, one can put a metric on the functions which take values in the regularity structure, generalizing the classical Hölder spaces, and solve the fixed point problem in the resulting space. Furthermore, one can build a *reconstruction operator* which pastes together the expansions at different points, so that the solution is realized as a genuine distribution, which, if the model  $\Pi^{(\varepsilon)}$  is the one we built from a smooth function  $\xi_\varepsilon$ , is just the classical solution to the original equation. A version of the renormalization group  $\mathfrak{R}$  acts on the set of admissible models, so if we can find  $M_\varepsilon \in \mathfrak{R}$  so that  $\Pi^{(\varepsilon)}M_\varepsilon$  converges, the limit is our answer. And the renormalized equations are just what we get by translating back through the  $M_\varepsilon$ . Now it turns out that in order to prove convergence one only has to check it on the terms of negative homogeneity, and local sub-criticality means that there are only a finite number of these multilinear transformations of the noise (super-renormalizability).

Thus the entire problem is reduced to finding the correct renormalization (usually by educated guesswork) and then proving the corresponding finite collection of multilinear transformations of the noise converges.

If the smooth approximating noises are Gaussian, we are in a situation of Gaussian chaos and there are easy criteria for convergence based on  $L^2$  norms. These can be very complicated generalized convolutions of the various kernels in the problem, and convergence depends on a careful counting of singularities. This convergence problem is highly reminiscent of earlier calculations in constructive quantum field theory, and a similar diagrammatic method is used to keep track of the cumbersome computations.

Up to this point we have only described some well-posedness results. But the method gives an analogue of the Itô calculus (the Hairer calculus!) for such equations, and an extremely powerful tool for approximations. For example, by adding appropriate operators to the KPZ regularity structure, one can prove [10] that for any even polynomial  $F$ , the  $-1 : 4 : 2$  scaling (3) of (12) converges to the standard quadratic KPZ equation

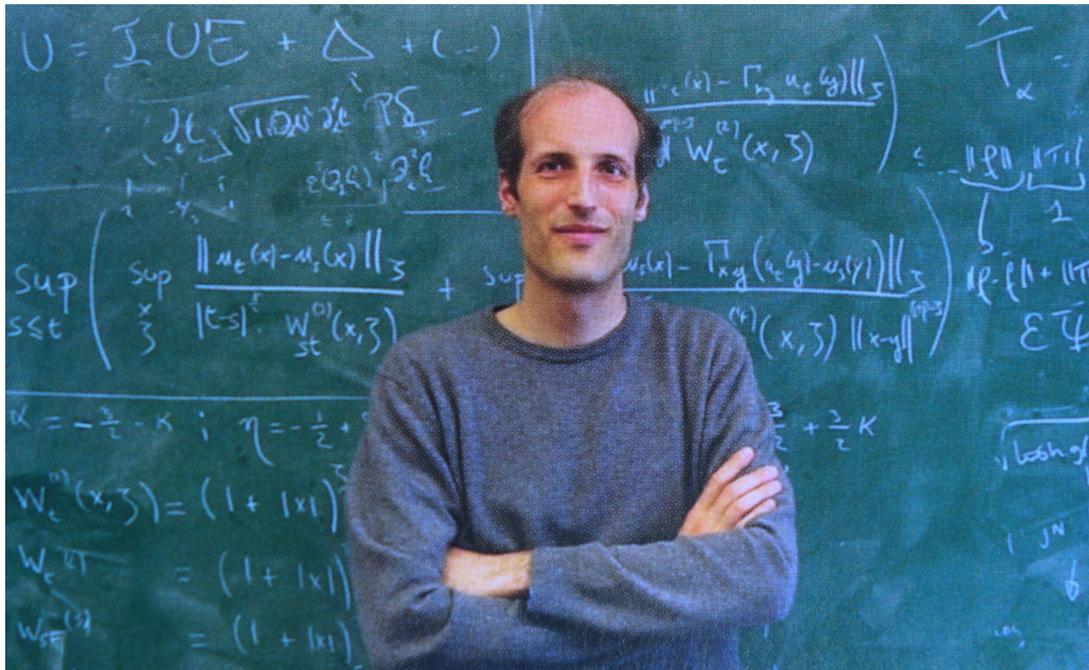
(6), but with a  $\lambda$  which is *not*  $\frac{1}{2}F''(0)$ , but

$$\lambda = \frac{1}{2}\partial_\rho^2 \bar{F}|_{\rho=0} \tag{16}$$

where  $\bar{F}(\rho)$  is the expectation of  $F$  with respect to a Gaussian with mean  $\rho$ .

The KPZ equation has a rich mathematical structure which has now led to a number of deep mathematical works. Besides Hairer's regularity structures, one should also mention the MacDonal Processes of Borodin and Corwin [1], part of a large scale attempt to get at the source of the integrability of the equation, and explain how the random matrix distributions are coming in. One looks forward to similar developments in the case of (7) and (8). The KPZ fixed point and stochastic Navier-Stokes equations also stand as highly non-trivial field theories awaiting our efforts.

Our heartfelt congratulations to Martin! We are already looking forward to your coming successes.



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# Mathematical Billiards and Chaos

by DOMOKOS SZÁSZ (Budapest)

## 1 Introduction.



On 20 May 2013, Yakov G. Sinai of Princeton University and of Landau Institute for Theoretical Physics, The Russian Academy of Sciences was awarded the Abel Prize at a ceremony at the University Aula in Oslo, [1]. In conjunction with Sinai's award there were four mathematical lectures, including the Prize Lecture by Sinai himself ("How everything has been started? The origin of deterministic chaos"), and distinguished mathematical lectures by Gregory Margulis of Yale University ("Kolmogorov-Sinai entropy and homogeneous dynamics") and Konstantin Khanin of University of Toronto ("Between mathematics and physics"). The present article is based on the Science Lecture talk I gave in Oslo having the same title. The videos of all four lectures can be viewed at <http://www.abelprize.no/artikkel/vis.html?tid=61307>.

My goal was to recite Sinai's most far-reaching results and their most significant after-effects on billiards which were distinguished in the Citation as follows: "Sinai has been at the forefront of ergodic theory. He proved the first ergodicity theorems for scattering billiards in the style of Boltzmann, work he continued with Bunimovich and Chernov. He constructed Markov partitions for systems defined by iterations of Anosov diffeomorphisms, which led to a series of outstanding works showing the power of symbolic dynamics to describe various classes of mixing systems."

By its definition *The Science Lecture is intended for the broader scientific community and aims to highlight connections between the work of the Abel Laureate and other sciences*. Moreover, the request of the Prize Committee emphasized as well that the Science Lecture is also meant for students of mathematics. To accomplish this task I will use many illustrations and metaphores.

Sinai's results on billiards have exceptional significance in the sciences. For an explanation I will go back to the birth of statistical physics.

## 2 Atomic theory and birth of statistical physics

Atomic theory assuming or claiming that matter is composed of discrete units called atoms or molecules goes back to thinkers from ancient Greece and India. Nevertheless extensive scientific theories relying on this assumption were only formed in the 19th century whereas – however surprising it may sound – its scientific confirmation only occurred in the 20th century. Sinai’s works mentioned in the citation originate from two fundamental questions raised by 19th century statistical physics.

The first puzzle is related to Ludwig Boltzmann’s ergodic hypothesis. Basic laws of statistical physics got formulated by classics of the theory: Maxwell, Gibbs, Boltzmann on a macroscopic level by assuming that, on the microscopic level, atomic motion obeys the laws of newtonian mechanics. In Boltzmann’s picture and in today’s language, macroscopic concepts of statistical physics: temperature, pressure, ... appear as the result of the law of large numbers valid in the microscopic system. His ergodic hypothesis claims that the equilibrium expected value of a measurement can also be obtained as the limit of time averages, under the newtonian dynamics, of the same measurement when both time and the size of the system tend to infinity (cf. [3, 4]).

The second problem also goes back to the 19th century. Due to advances in microscopes inquisitive scholars could observe an increasingly wider circle of phenomena under them. For instance, in 1827 Robert Brown, a British botanist, inspected under his instrument pollens of the plant *Clarkia pulchella* suspended in water. To his great surprise pieces of pollen showed an erratic, random motion. He got even more astonished by probing minerals including liquids, isolated for millions of years: tiny particles suspended in them displayed the same erratic motion. Is it imaginable that living substance survives in the inclusion for such a long time? Or can this chaotic motion be explained by atomic theory? Is it the result of collisions of the observed ‘Brownian’ particle with atoms of the liquid?

This latter idea got quantified in 1905 in the derivation of the diffusion equation by Einstein, a believer in atomic theory. The diffusion equation is, indeed, the appropriate model for the chaotic motion of the Brownian particle and likewise it is worth noting that the same equation also depicts the flow of heat in many substances. It is less known that – motivated by the fluctuations of stock prices – Louis Bachelier, as early as in 1900, already introduced the stochastic process solving the diffusion equation and also describing Brownian motion. (Einstein does not seem to have known about this work.) The precise mathematical definition of this process was provided in the early 1920s by Norbert Wiener and most often it is called a Wiener process (or Brownian motion process).

## 3 Ergodic Hypotheses: from Boltzmann to Sinai

In today’s language the basic object of ergodic theory is a group  $(S^{t+s} = S^t S^s, t, s \in \mathbb{R})$  of measure preserving transformations  $\{S^t | t \in \mathbb{R}\}$  on some probability space  $(M, \mathcal{F}, \mu)$ . Here the non-empty set  $M$  is the phase space and  $\mu$  is a probability measure on it. Measure preserving means that the dynamics  $\{S^t\}$  leaves the probability  $\mu$  invariant, i.

e.  $\forall A \in \mathcal{F}$  and  $t \in \mathbb{R}$  one has  $\mu(S^t A) = \mu(A)$ . (Later we will forget about the  $\sigma$ -algebra  $\mathcal{F}$ .)

For Boltzmann's formulation of the ergodic hypothesis consider  $N$  particles in a vessel that – for simplicity – can be the torus  $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$  (or the cube  $[0, 1]^3$ ). Thus the phase space of the system is  $M_N = \{(q_i, v_i) | 1 \leq i \leq N, q_i \in \mathbb{T}^3, v_i \in \mathbb{R}^3, \sum v_i^2 = 1\}$ . Of course, the dynamics  $S_N^t x_N$  ( $t \in \mathbb{R}$  – time,  $x_N \in M_N$ ), the invariant probability  $\mu_N$  and the measurement  $f_N : M_N \rightarrow M_N$  all depend on  $N$ .

**Conjecture 1 (Boltzmann's Ergodic Hypothesis)** *For a typical initial phase point  $x \in M$*

$$1/T \int_0^T f_N(S_N^t x) dt \rightarrow \int_{M_N} f_N(y) d\mu_N(y)$$

*as  $T, N \rightarrow \infty$  (or in other words time averages converge to the equilibrium average).*

This is a hypothesis of an ingenious theoretical physicist. It is far from having a precise mathematical meaning (for instance, neither the sense of convergence of time averages nor the way  $T$  and  $N$  tend to  $\infty$  is specified) though it is known that Boltzmann had, indeed, made a lot of tricky calculations to convince at least himself that the conjecture holds. (According to Boltzmann's picture a physical system in equilibrium – in its time evolution – 1.) not only goes through all possible phase points of the system but it does that in such a way that 2.) the relative sojourn times in subsets are close to the equilibrium law of the system. Claim 2.) is just the content of his ergodic hypothesis whereas claim 1.), in general, fails.) Despite of this deficit of exactness, this conjecture and likewise the derivations of his famous equation or his H-theorem and other's statistical arguments were known to several top mathematicians of the era. In so much that David Hilbert – in his celebrated lecture on ICM1900 about 23 problems of mathematics for the 20th century – included one, notably the 6th one with the title *Mathematical Treatment of Axioms of Physics*. Its issue was to give a precise mathematical background to related arguments of theoretical physicists and Hilbert himself said "... I refer to the writings of Mach, Hertz, Boltzmann and Volkmann...".

The response of mathematics was not immediate. Having been inspired by many sources in the first decade of 20th century, modern measure theory was founded first and later in the 20s – pressed by the necessity to provide mathematical framework for quantum physics then in statu nascendi – functional analysis and the theory of operators were created. In the academic year 1926/27 John von Neumann visited Hilbert in Göttingen. Until this visit his main interest was pure mathematics and the reason for the visit was his shared interest with Hilbert in the foundations of set theory. Nevertheless during this year von Neumann got deeply involved in operator theory, which certainly helped him to understand the mechanism behind the convergence of time averages, and he proved the first ever ergodic theorem. Consider a fixed dynamical system  $(M, S^t, \mu)$  as formulated at the beginning of this section.

**Definition 1**  *$(M, S^t, \mu)$  is ergodic if  $\forall x \in M, \forall t \in \mathbb{R} f(S^t x) = f(x)$  implies  $f = \text{const}$  for  $\mu$ -a.e.  $x \in M$ . (In other words, every invariant function is constant almost everywhere.)*

For an illustration let us try to decide whether the simplest 'physical' dynamics: uniform motion on the euclidean 2-torus is ergodic.

**Example 1: Geodesic motion on the (euclidean) 2-torus  $\mathbb{T}^2$ .**

$M = \mathbb{T}^2 \times \mathcal{S}$ , ( $\mathcal{S} = \mathbb{T}^1$ ). For  $x = (q, v) \in M$   $S^t x = (q + tv, v)$ ,  $\mu = \text{area}$ . We note that the restriction  $\|v\| = 1$  reflects the conservation of energy. This 'boring' dynamics is, of course, nonergodic since  $f(q, v) = v$  is a non-constant invariant function.

**Theorem 1 (Neumann's Ergodic Theorem, 1931)** *Assume  $(M, S^t, \mu)$  is ergodic and  $f : M \rightarrow \mathbb{R}$  is a nice function. Then, as  $T \rightarrow \infty$ ,*

$$1/T \int_0^T f(S^t x) dt \xrightarrow{L_2} \int_M f(y) d\mu(y) \quad (*)$$

For instance, if – in an ergodic system – one takes the indicator function of a set  $A$ :

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases},$$

then according to the ergodic theorem the average time spent by the system in the set  $A$  converges to the probability  $\mu(A)$  of the set as  $T \rightarrow \infty$ . (Later, George Birkhoff (1931) and Khinchin (1933) proved that (\*) also holds for  $\mu$ -a. e.  $x \in M$ .)

A survey, written in 1932 by Birkhoff and Koopmans about the history of the first ergodic theorems wrapped up : "It may be stated in conclusion that *the outstanding unsolved problem in the ergodic theory* is the question of the truth or falsity of *ergodicity for general Newtonian systems*." Indeed, since the search for ergodic theorems was motivated by physics, it is inevitably a major question of ergodic theory to clarify the mechanism lying behind the ergodic behavior of physical motions. Once uniform motion in a euclidean domain was not ergodic it was a natural idea to investigate uniform – or in other words geodesic – motion in a hyperbolic domain.

**Example 2: Geodesic motion on the hyperbolic octagon.**

Let us consider the Poincaré disk model of hyperbolic (i. e. Bolyai-Lobatchevsky) geometry:  $D = \{z \mid |z| < 1\}$ . At a point  $z \in D$  the hyperbolic metric  $(ds)_{\text{hyp}}$  can be expressed through the euclidean metric  $(ds)_{\text{euc}}$  as follows:  $((ds)_{\text{hyp}})^2 = \frac{((ds)_{\text{euc}})^2}{(1-r^2)^2}$  where  $r = |z|$ . As is well-known,  $D$  supplied with this metric is a surface of constant negative curvature. In this case the phase space is  $M_\infty = \{x = (q, v) \mid q \in D, v \in \mathbb{R}^2, |v| = 1\} = D \times \mathcal{S}$ . The orbits of uniform, i. e. of geodesic, motion are circular arcs orthogonal to the circle  $\partial D$  of infinities (see Fig. 1). Completely analogously to the euclidian case, also here one should factorize wrt some discrete subgroup (there it was  $\mathbb{Z}^2$ ). Here too, there is an abundance of possibilities and we select one of the simplest cases: we obtain a hyperbolic octagon  $O$  and it is compact and the invariant measure is finite (see Fig. 2).

**Theorem 2 (Hedlund, Hopf (1939))** *The geodesic motion in the hyperbolic octagon (i. e. on  $M = O \times S$ ) is ergodic.*

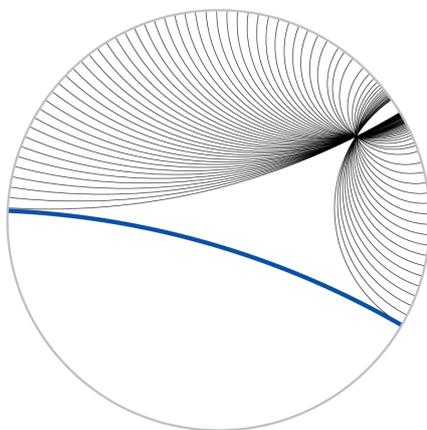


Figure 1: Geodesics in the hyperbolic plane

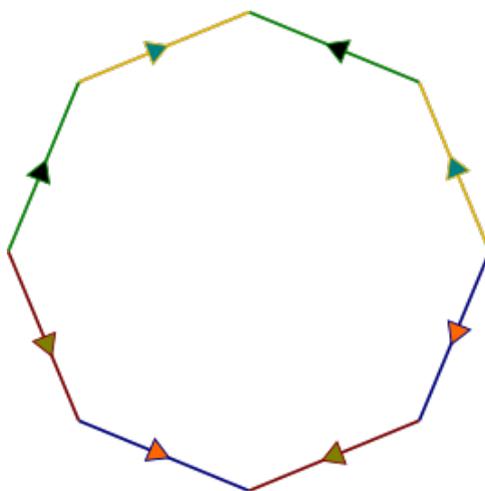


Figure 2: A hyperbolic octagon

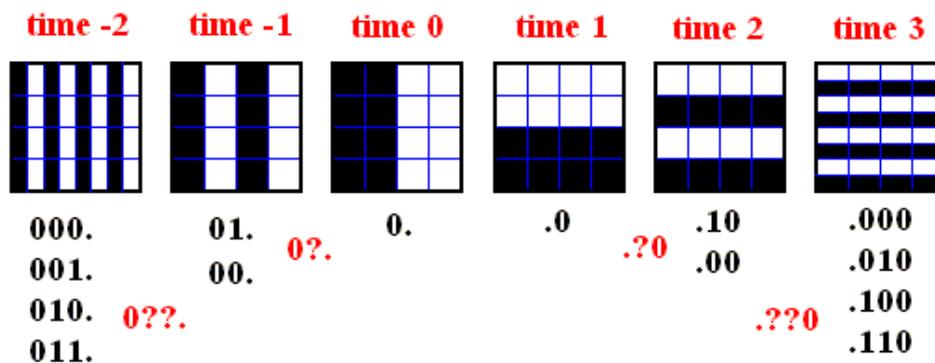


Figure 3: Iterates -2, -1, 0, 1, 2 of the baker map

The value of this result is not only that it represented the first system of physical flavour whose ergodicity was ever established but it also revealed the mechanism that can lead to ergodicity: it was profoundly connected to the hyperbolic feature of the geometry. The hyperbolicity of the dynamics can be easily illustrated by a favourite paradigm of ergodic theory courses: the baker map  $T : [0, 1]^2 \rightarrow [0, 1]^2$  (see Fig. 3).  $T$  is defined as follows:

$$T(x, y) = \begin{cases} (2x, y/2) & \text{if } 0 \leq x \leq 1/2, \\ (2x - 1, \frac{y+1}{2}) & \text{if } 1/2 < x < 1. \end{cases}$$

On the figure one can easily see that – at each step – in the horizontal direction the map expands by a factor of 2 and in the vertical one it contracts by the same factor of 2 and thus the area remains invariant. The composed effect of expansion and contraction is just hyperbolicity.

The proof also relied on constructions of Hadamard and introduced the method of Hopf chains that is still essentially the only general method for establishing ergodicity of hyperbolic systems.

Not only many top mathematicians but often top theoretical physicists have also been aware what interesting is going on in each other’s science. Thus in 1942 N. S. Krylov, an excellent Russian statistical physicist was playing around with the ideas of Hedlund and Hopf. His calculations led him to a remarkable observation: at collisions of elastic hard ball particles some hyperbolicity seem to appear.

In the 60s there was a remarkable progress in the theory of the smooth hyperbolic dynamical systems. Here the main names to be mentioned were Anosov, Sinai and Smale. These systems provide far-reaching generalizations of geodesic flows on compact manifolds of negative curvature. On ICM1962 Sinai, combining Krylov’s observation and the momentum of this progress, came up with the sensational conjecture:

**Conjecture 2 (Sinai’s form of Boltzmann’s ergodic hypothesis, 1962)** *A system of  $N(\geq 2)$  identical elastic hard balls on  $\mathbb{T}^\nu$  ( $\nu \geq 2$ ) is ergodic (modulo the natural in-*

*variants of motion: energy, momentum, center of mass that are, of course, invariant functions).*

This conjecture is fully in line with the program formulated by Birkhoff and Koopman. Nevertheless, at that time it deeply surprised experts who had been thinking that – in accordance with Boltzmann – for having ergodicity the number of particles should tend to infinity. According to Sinai already two hard balls should produce ergodicity?! Now we know that this is the peculiarity of the hard core interaction of balls. (For typical interactions the system of a fixed number of particles will not be ergodic since KAM islands will appear. What one can hope for is that: the larger the number of particles, the closer to 1 the measure of the largest component is.) The reader interested the early history – until 1996 – of the Boltzmann-Sinai ergodic hypothesis finds more details in [16].

## 4 Sinai Billiards. Verification of the Boltzmann-Sinai Hypothesis

### 4.1 Sinai, 1970 and Case $N = \nu = 2$

The difficulty of the verification of Sinai’s conjecture, called today the Boltzmann-Sinai ergodic hypothesis (or briefly BSEH), is shown by the fact that Sinai needed eight years to complete the proof for the ‘simplest’ case of two planar – not too small – disks (cf. [12]). His approach used the introduction of dispersing billiards, also called Sinai-billiards.

**Definition 2** *A billiard is a dynamical system  $(M, \{S^t | t \in \mathbb{R}\}, \mu)$  where  $M = Q \times S_{d-1}$ ,  $Q$  is a domain in  $\mathbb{R}^d$  or  $\mathbb{T}^d$  with piecewise smooth boundary,  $\mu$  is the uniform probability measure on  $M$  and  $\{S^t | t \in \mathbb{R}\}$  is the billiard dynamics: uniform motion in  $Q$  with velocity  $\in S_{d-1}$  and elastic reflection at  $\partial Q$ . Smooth pieces of  $\partial Q$  are called scatterers. (Fig. 4 illustrates a Sinai billiard on the 2-torus.)*

Billiards had already been used before Sinai, for instance planar ones in an oval by Birkhoff. The reason for Sinai’s introduction of dispersing like billiards was based on the following simple

**Fact.** *The system of  $N \geq 2$  elastically colliding identical hard balls on  $\mathbb{T}^\nu$  is isomorphic to a billiard whose scatterers are (subsets of) spheres when  $N = 2$  and (subsets of) cylinders when  $N > 2$ .*

This isomorphic billiard can be obtained by first putting together all the centers of the individual particles into an  $N\nu$ -dimensional vector. Because of the hard core condition no pair of centers can get closer than  $2R$  where  $R$  denotes the radii of the balls. These excluded sets are cylinders with spherical bases. Now by taking into account the conservation laws, the dimension of the ‘billiard table’ gets reduced by  $\nu$  (total momentum and ‘center of mass’ are assumed to be zero). Thus the dimension of the isomorphic billiard is  $d = (N - 1)\nu$ .

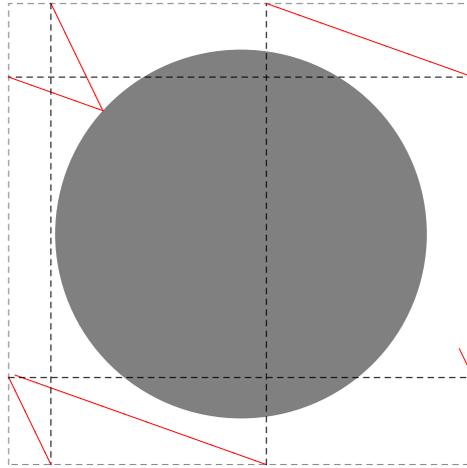


Figure 4: Orbit of a Sinai billiard (with one circular scatterer) on the 2-torus

**Definition 3** *A billiard is semi-dispersing (dispersing) if the scatterers are convex (strictly convex, respectively) as seen from the billiard table  $Q$ . Dispersing billiards are also called Sinai-billiards.*

Thus the system of two balls is isomorphic to a dispersing billiard whereas that of  $N \geq 3$  balls to a semi-dispersing one. For instance, on Fig. 5 one can see the isomorphic billiards corresponding to the case  $N = \nu = 2$  (cases  $R < 1/4$  resp.  $R > 1/4$ ).

**Definition 4** *A billiard is said to have finite horizon if there is no infinite orbit without any collision. Otherwise it has infinite horizon. (For instance, in Fig. 5 the first billiard has infinite horizon whereas the second one has finite one.)*

Sinai's celebrated 1970 result was:

**Theorem 3** ([12]) *Every planar dispersing billiard with finite horizon is ergodic. (In particular, the system of two elastic disks is ergodic if  $R > 1/4$ .)*

Later we will illustrate why dispersing billiards are hyperbolic but first we reveal a fundamental difficulty arising in billiards: the singularity of the dynamics. For both purposes it is useful to consider the evolution of fronts: smooth one-codimensional submanifolds  $\Sigma$  in the configuration space  $Q$  lifted to the phase space by adjoining to each point  $q \in \Sigma$  an orthogonal unit normal vector  $\in S_{d-1}$  as a velocity vector (this procedure has two continuous versions; later we will be interested in convex fronts, where the curvature operator of  $\Sigma$  is non-negative definite and then we will assume that the attached normal vectors open up). Indeed, as it is shown on Fig. 6, in case of a tangential collision of a front its image is not smooth any more; in fact, after the collision its image is still continuous but is not differentiable any more. Moreover, the reader can convince himself/herself that if the orbit hits an intersection point of different scatterers, then afterwards the dynamics is not uniquely defined, the trajectory will have at least two smooth branches.

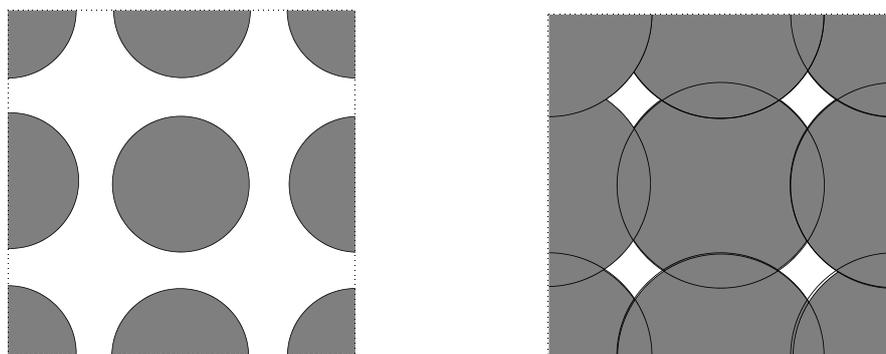


Figure 5: Sinai billiard isomorphic to dynamics of two elastic disks

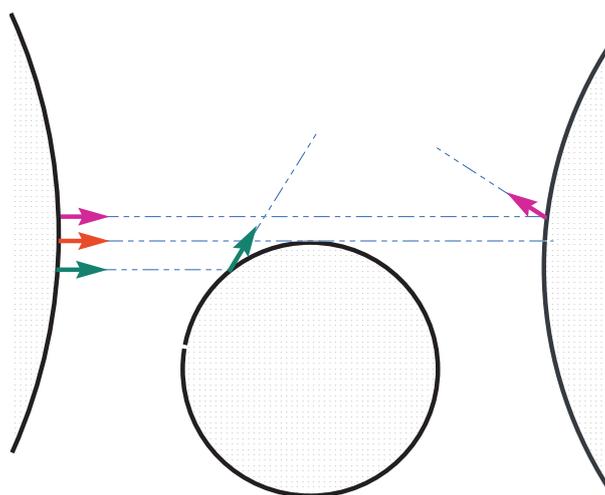


Figure 6: Tangential singularity of a billiard

In view of the singularity of the dynamics, the main achievement of Sinai's 1970 work was to show that *the theory of smooth hyperbolic dynamical systems can be extended to that of hyperbolic systems with singularities – at least in 2D*. This extension may seem to be a harmless generalization but, in fact, it is a wonder that it is possible at all. From Hadamard, Hedlund and Hopf, and later from Anosov, Sinai and Smale one had known that the main tools of the hyperbolic theory were the so called unstable (and stable) invariant manifolds possessing two important properties: 1) they were smooth and 2) the time evolution of the unstable ones had an exponentially expanding effect (in case of the stable manifolds their time reversed evolution had it; cf. the baker map of Fig. 3 considered earlier). Sinai could show that, in case of singular systems, smooth pieces of invariant manifolds still exist for almost every point, they can, however, be arbitrarily small. This fact had made the application of classical techniques though possible but still a rather hard task. Sinai could cope with this and it is also worth mentioning that the style of his work was also optimal: though the subject was new he avoided overexplaining ideas; it provided as much information as much was needed.

His forthcoming paper with Bunimovich could drop the condition on the finiteness of the horizon and with this it completed the verification of the BSEH for two disks.

**Theorem 4** ([7]) *Every planar dispersing billiard with infinite horizon is ergodic. In particular, two elastic disks on the two-torus is ergodic.*

## 4.2 Chernov-Sinai, 1987 and Case $N = 2, \nu \geq 2$

To proceed with the proof of the BSEH there were two tasks: to also treat 1) the multi-dimensional case; 2) the semi-dispersing case. I have promised to go back to the question why these dispersing type billiards are hyperbolic. This is based on the following intuitively transparent facts:

**Fact 1:** For a dispersing billiard – after one collision – any convex front becomes strictly convex.

**Fact 2:** For a semi-dispersing billiard – convexity of a front can only improve.

These facts are illustrated on Fig. 7 and 8. Indeed, Fig. 7 should convince the reader that after a collision with a strictly convex scatterer even a hyperplanar front, the extreme case of a convex one, will become strictly convex. On the other hand, Fig. 8 shows that after a collision with a cylindrical scatterer, the direction(s) of a hyperplanar front, parallel with the generator of the cylinder will remain linear implying that the image of the hyperplanar front will not be strictly convex.

If we accept that for hyperbolicity strict convexity of the image of any incoming convex front is necessary, then what we can hope for is that this at least happens after collisions with several different scatterers. This explains the following fundamental

**Definition 5** *A phase point  $x = (q, v) \in M$  is a hyperbolic point if the hyperplanar front through  $q$  and orthogonal to  $v$  becomes strictly convex – maybe only after several collisions (cf. Fig. 9).*

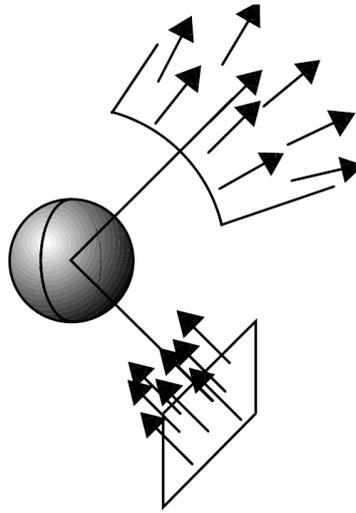


Figure 7: Collision in a dispersing billiard

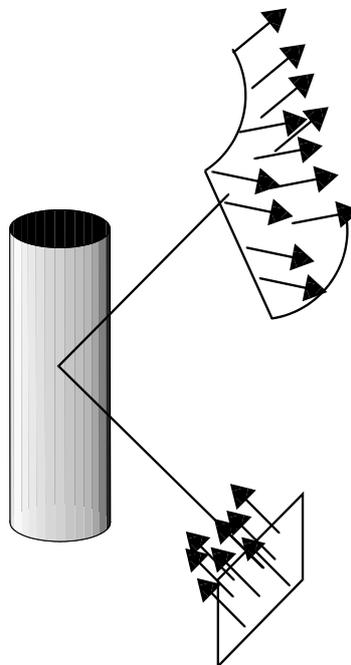


Figure 8: Collision in a semi-dispersing billiard

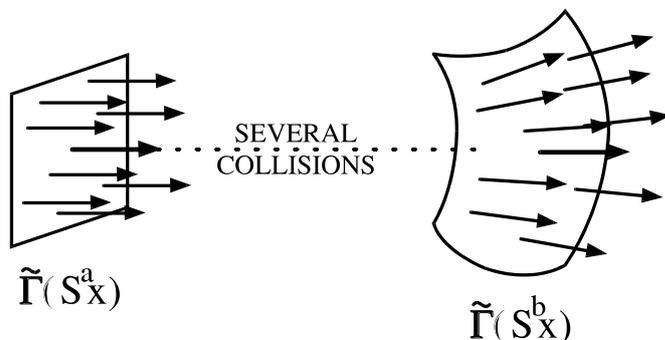


Figure 9: Hyperbolic orbit

**Definition 6** A subset  $U \subset M$  belongs to one ergodic component if  $\forall x \in U, \forall t \in \mathbb{R} f(S^t x) = f(x)$  implies  $f = \text{const}$  for  $\mu - \text{a.e. } x \in U$ . (Compare this with Definition 1.)

Now we are in the position to formulate the main result of a beautiful work of Chernov and Sinai, born after 17 years from Sinai's classical paper.

**Theorem 5 ([11])** For a semi-dispersing billiard ( $d \geq 2!$ ), under some additional conditions every hyperbolic point has an open neighborhood which belongs to one ergodic component.

A simple but important corollary of this theorem is:

**Corollary 1 (simple)**

1. Every dispersing billiard is ergodic ( $d \geq 2$ ).
2. The system of TWO hard balls on any  $\mathbb{T}^\nu$  is ergodic.

**4.3 Case  $N \geq 2, \nu \geq 2$**

Here we are satisfied to just list some main steps leading to the complete verification of the BSEH by Simányi in 2013.

- Going from the case  $N = 2$  to the case  $N > 2$  required the solution of additional topological, ergodic-theoretical and algebraic problems. In [10] this was done for the case  $N = 3, \nu \geq 2$  of the BSEH.
- [6] found a model of hard balls with  $N \geq 2, \nu \geq 2$  whose ergodicity could be shown. The technical ease of the model is that the balls are localised in it and as a result only balls in neighbouring cells can collide.

- By introducing algebraic geometric methods into the problem, [15] could prove that, for any  $N \geq 2, \nu \geq 2$ , typical systems of hard balls are hyperbolic.
- Though hyperbolicity is actually the antechamber of ergodicity, there still remained a lot of work to be done until the complete verification of the BSEH. Simányi pursued this goal in a series of delicate works, until he could finish the arguments in [14].

## 5 Dynamical theory of Brownian motion

In section 2 I have already referred to Brown's discovery of the chaotic motion of particles suspended in a fluid. Einstein's derivation for it from microscopic assumptions was heuristic and used conditions which were not satisfied. In fact, one of his main aims with this work was to derive an estimate for the Avogadro number on the basis of atomic theory. This was so successful that it inspired the French experimental physicist, Jean Baptiste Perrin, to experimentally test Einstein's calculation which he did in 1908. His observations were in full agreement with Einstein's calculations and this finally led to a complete acknowledgment of atomic theory (for this work Perrin was deservedly awarded the Nobel Prize in Physics in 1926).

Since Einstein's explanation of the Brownian motion was heuristic, the question was still left completely open to provide a microscopic and mathematically rigorous derivation for it. An intermediate step came first from probability theory. Random walks have been a classical and favourite model of stochastics. The simple symmetric random walk (SSRW) is a discrete time Markov process  $\{S_n, n \in \mathbb{Z}_+\}$  on  $\mathbb{Z}^d$ ;  $d \geq 1$  such that  $\forall n \in \mathbb{Z}_+, \forall k \in \mathbb{Z}^d \mathbb{P}(S_{n+1} = k \pm e_j | S_n = k) = \frac{1}{2d}$ , where  $e_j$  is any of the  $d$  unit coordinate vectors in  $\mathbb{R}^d$ . For making the ideas transparent we present several figures: Fig. 10 shows several trajectories of 1D SSRWs, Fig. 11 one orbit of a 1D Brownian motion. Fig. 12-14 present orbits of 2D SSRWs for 2500, 25000, 250000 steps, respectively. Finally Fig. 15 shows an orbit of a 2D Brownian motion. These figures suggest that – independently of the dimension  $d$  – an orbit of a Brownian motion arises if one looks at an orbit of a SSRW from a distance.

The mathematical formulation of this claim is the following: take a large parameter  $A \in \mathbb{R}_+$  and rescale time with  $A$  and space with  $\sqrt{A}$  (this is in accordance with the fact that the variance of  $S_n$  is  $\sqrt{n}$ ). In other words consider the rescaled trajectory  $X_A(t) = \frac{S_{At}}{\sqrt{A}}; t \in \mathbb{R}_+$  of the SSRW (so far  $S_{At}$  has only been defined for integer values of  $At$ ; we overcome this deficiency by taking the piecewise linear extension). Then a classical and beautiful result of probability theory says that  $X_A(t) \Rightarrow W(t)$  as  $A \rightarrow \infty$  where  $W(t)$  is the Wiener process. (The convergence is understood in the weak sense of probability measures.) The meaning of this claim is the following: one starts with a SSRW, a stochastic model of the motion of an individual particle and then one obtains the Wiener process, the model of Brownian motion in an appropriate scaling. The true result would be an analogous statement for a deterministic model. In probability theory and in statistical physics there had been a lot of efforts to try and find such models.

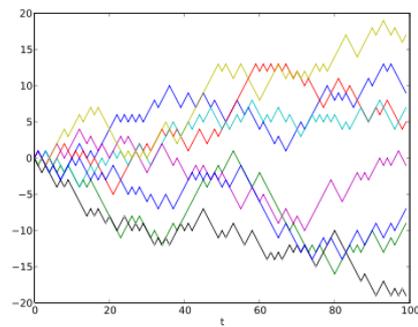


Figure 10: Trajectories of a 1D Random Walk

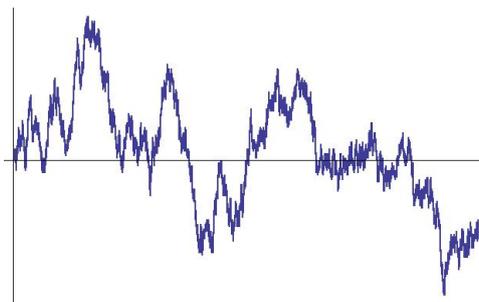


Figure 11: Orbit of a 1D Brownian motion

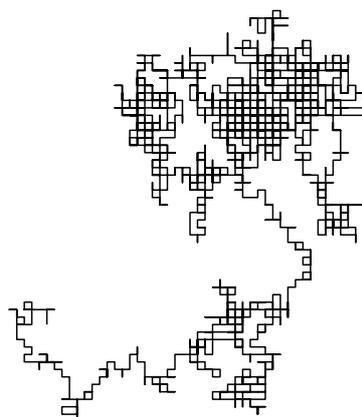


Figure 12: Orbit of a 2D Random Walk (2500 steps)

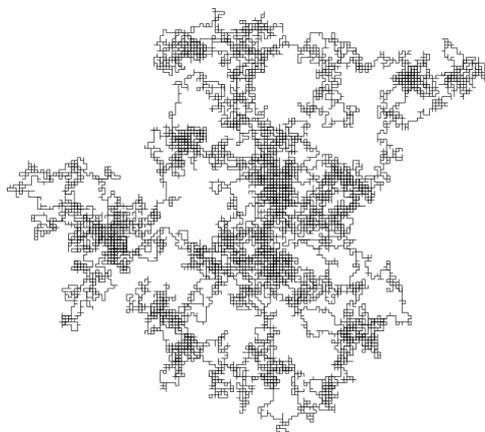


Figure 13: Orbit of a 2D RW (25000 steps)

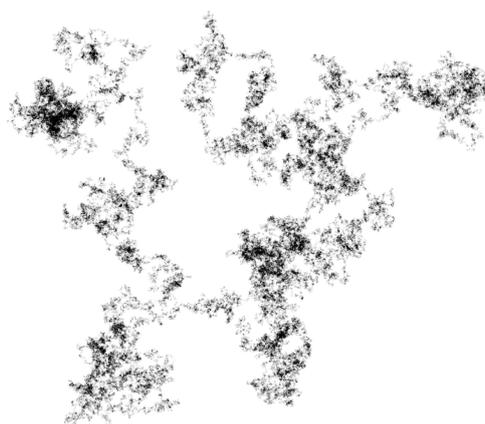


Figure 14: Orbit of a 2D RW (250000 steps)



Figure 15: Orbit of a 2D Brownian motion

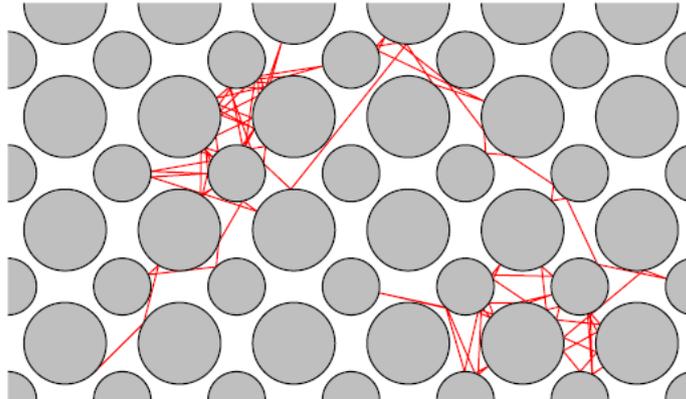


Figure 16: Orbit of a 2D Lorentz process (finite horizon)

Sinai, with Bunimovich, later including also Chernov, [8, 5] considered a deterministic model of Brownian motion introduced in 1905 by the Dutch physicist Hendrik Antoon Lorentz. He designed the so called Lorentz process to describe the motion of a classical electron in a crystal. (The Nobel Prize winner Lorentz is widely known for the Lorentz transformation of relativity theory.) Mathematically speaking the periodic Lorentz process is a  $\mathbb{Z}^d$ -extension of a Sinai billiard and Fig. 16 illustrates its finite horizon version for  $d = 2$ . We emphasize that it is a deterministic motion  $L(t)$ , in fact it is a billiard trajectory on the plane with only the initial point  $L(0)$  selected at random. By using the same scaling which we used for the SSRW let us define  $L_A(t) = \frac{L(At)}{\sqrt{A}}$ . Then

**Theorem 6** ([8, 5]) *Let  $L(t)$  be a planar finite horizon Lorentz process with periodic scatterers. Then*

$$L_A(t) \Rightarrow W(t) \quad \text{as } A \rightarrow \infty$$

I mentioned that the periodic Lorentz process is an extension of the Sinai billiard. The proof of ergodicity of the planar Sinai billiard was quite a breakthrough, but – even after its proof – proving the above theorem required at least as much creative effort than the proof of ergodicity. One more remark: some of Sinai’s achievements, like the proof of ergodicity and the theorem above, are solutions of outstanding, utmost important problems. However, the methods for the above theorem were prepared by Sinai’s theory-building activity also emphasized in the part of the Citation at the start of this paper. In fact, starting from 1968, Sinai was deeply involved in the construction and applications of Markov partitions for smooth hyperbolic systems. Here we only circumscribe the essential philosophy of this fundamental activity. Once one can construct a Markov partition, satisfying some good properties, for a dynamical system, then instead of the original dynamical system one can investigate the isomorphic symbolic system. This later one is not a Markov chain but often it can be efficiently approximated by higher order

Markov chains of longer and longer memory. These are, however, objects of probability theory permitting the applications of its artillery for the understanding of the dynamical behaviour of the system. This philosophy culminated in Sinai's work [13]. The works [8, 5] were based on constructing Markov partitions (or weaker and more flexible objects, called Markov sieves) for planar Sinai-billiards.

As it happened with ergodicity, the construction of effective Markov approximations for planar billiards raised the demand and opened the way for multidimensional extensions. This is still beyond reach, yet there exist more flexible variants of Markov approximations. We mention two important and promising ones:

1. LS Young, [18] introduced the method of Markov towers also providing a new proof for the previous theorem of Sinai and coauthors. The method is also applicable to treat stochastic properties of planar singular hyperbolic systems, in general. Thus, for instance, it also works for dissipative maps, like the Hénon-map, or unimodal maps of the interval. Another advantage of this method was that it permitted to obtain optimal, exponential bounds for the speed of correlation decay, i. e. for the relaxation to equilibrium. An important generalization of the tower method was given by Bálint and Tóth, [2], where Markov towers were constructed for multidimensional finite horizon dispersing billiards under an additional 'complexity condition'. Further, by the application of Young's tower method Szász and Varjú, [17] could extend the aforementioned theorem of Sinai and coauthors to the infinite horizon case by showing that in this case

$$\frac{L(At)}{\sqrt{A \log A}} \Rightarrow W(t) \quad \text{as } A \rightarrow \infty$$

2. In [9] Chernov and Dolgopyat made the method of Markov approximations even more flexible by introducing the method of standard pairs. It also permitted them to extend the circle of mechanical models where variants of Brownian motion can be derived.

In section 1 I explained which classical significant problems of statistical physics billiards – and the Lorentz process based upon it – help to understand and hopefully answer. In fact, billiard models seem to be most appropriate for this purpose. For conclusion I note that the derivation of laws of statistical physics from microscopic principles is still a fundamental goal of mathematicians and physicists. I have only mentioned here the ergodic hypothesis and the dynamical theory of Brownian motion. This approach has had a lot of successes and by the progress of mathematical techniques, which owe their birth to Sinai, it is able to cover a wider and wider circles of phenomena (recently much effort is focussed on understanding heat transport).

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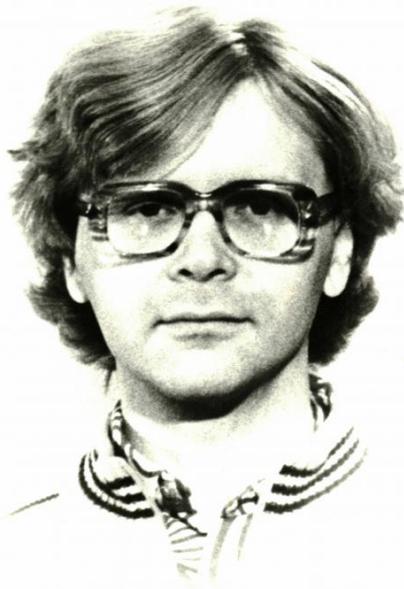
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## **Kolya Chernov**

**(1956–2014)**



The Mathematics and Mathematical Physics communities have suffered a heavy and untimely loss. After a long fight with cancer Professor Nikolai Ivanovich (Kolya) Chernov passed away on the seventh of August 2014. Kolya was born and grew up in the city of Kryvyi Rih in the Ukraine (former USSR). His interest in Mathematics appeared early and began to flourish already in Kolya's school years. Kolya Chernov was a winner of the State, the Republic and the International Mathematical Olympiad. In 1974 Kolya entered the Department of Mechanics and Mathematics of the Moscow State University. He had chosen the field of Probability Theory and started to work under the guidance of Professor Sinai. Kolya graduated at 1979 with a Master Degree. His Master thesis was devoted to the study of kneading invariants of one-dimensional maps with an absolutely continuous invariant measure. It was the topic of the first paper out of more than a hundred research papers published by Professor Chernov. In the same year Kolya became a graduate student in the Department of Mechanics and Mathematics continuing his work with Prof. Sinai. At that time he became interested in the dynamics of chaotic billiards which remained the love of his scientific life. The theory of chaotic billiards forms one of the most exciting and the most difficult areas in the modern theory of dynamical

systems. A billiard dynamics is generated by the motion of a point particle with constant speed in a region  $Q$ , which is called a billiard table. Upon reaching the boundary of  $Q$  the particle gets elastically reflected according to the law “the angle of incidence equals the angle of reflection.” Billiards appear as natural models in many areas of physics, above all in statistical mechanics. The study of billiards becomes especially difficult when the dimension of the billiard table is greater than two. Kolya’s PhD thesis, which he defended in 1984, was devoted exactly to such billiards. In particular, he constructed the stable and unstable manifolds for multidimensional semi-dispersing billiards. After defending his PhD, Dr. Chernov was working as a research scientist in the Joint Institute for Nuclear research in Dubna. As a part of the team in the Information Technology lab he was working on statistical analysis of data from physics experiments. A problem they faced was how to approximate noisy data with simple curves like straight segments and circles. It was the main characteristic of Kolya’s scientific style to penetrate deeply into any problem he was dealing with. It is no surprise that he became an expert in this field as well. He published many papers about this topic and several of his students in the University of Birmingham defended PhDs in this area. Kolya also published a very well received book, “Circular and Linear Regression: Fitting Circles and Lines by Least Squares”, (2010), where he summarized the geometric, algebraic and numerical aspects of fitting lines and circles to data. However billiards were always the greatest of Prof. Chernov’s scientific interests. Dubna is close to Moscow, and therefore he could continue the seminar on Dynamical systems in Moscow university and collaborate with Prof. Sinai. The most important example of a high-dimensional billiard is the celebrated Boltzmann gas of  $N$  hard spheres confined in a box (or in a torus). This system is equivalent to a semi-dispersing billiard, where the boundary of the billiard table consists of cylinders. Each such cylinder corresponds to the collisions of some fixed pair of particles. The fundamental problem of ergodicity of the Boltzmann gas with two particles was settled by Sinai. However in dimensions greater than two the situation becomes technically much more complicated. In Kolya’s joint paper with Sinai (1987) a general scheme was presented to prove ergodicity of the Boltzmann gas. This paper contains the celebrated Sinai-Chernov Ansatz which became a cornerstone for proving ergodicity of non-uniformly hyperbolic dynamical systems. Chaotic billiards belong to this fundamental class of dynamical systems because of singularities which arise from the tangent collisions and/or singularities of the boundary of the billiard table. In 1991 Kolya moved to the US. He held visiting positions at UCLA, Georgia Tech and Princeton before joining the Mathematics Department in the University of Alabama in Birmingham at 1994. There Prof. Chernov was actively working with a large number of undergraduate and graduate students. Some of them became visible researchers in the theory of billiards. Already in 1991 Kolya started a new area of research. Together with Eyink, Lebowitz and Sinai they constructed nonequilibrium steady states and investigated ergodic and statistical properties of the Lorentz gas with a small electric field. Because the speed of the moving particle in this system goes with time to infinity, one needs to introduce some kind of dissipation. This was done with the help of the so called Gaussian thermostat. Kolya continued to develop this important area with the same collaborators and with his

students till the very end. Perhaps the most important and groundbreaking of Kolya's papers was the one published in 1998 where he proved a quasiexponential rate of decay of correlations for Anosov flows. It was well known that Anosov diffeomorphisms have an exponential rate of decay of correlations. However in dynamical systems with continuous time (flows) there appears a new (time) dimension and along this dimension there is no exponential instability which one has along the stable and unstable transversal manifolds. It was the first dynamical proof of the fast decay of correlations in systems with continuous time which was based on new insights and a very delicate new technique. Before this such a result was known only for geodesic flows on compact manifolds with negative curvature. However the proof of this fact was purely algebraic and did not shed any light on the dynamical mechanism behind this phenomenon. In 2002 together with Balint, Szasz, and Toth, Kolya discovered new types of singularities in the stable and unstable manifolds of semi-dispersing billiards, which was a fundamental result for the theory of such billiards. Kolya established a very fruitful collaboration with Dima Dolgopyat. They produced several remarkable papers which solved some long standing classical problems. Perhaps the most impressive of their results is a two hundred page proof of the existence and limiting distributions of the classical model of real diffusion, where a heavy particle elastically collides with a gas of noninteracting light particles. This is exactly a model of Brownian motion, as observed by Brown. For almost forty years the theory of chaotic billiards was virtually inaccessible for beginners. It was contained in research papers which were very hard to read. Together with Markarian, Kolya beautifully filled in this gap by producing the book "Chaotic Billiards" (2006) which has already made a great impact and allowed many gifted young mathematicians to join in the exciting game of mathematical billiards. Kolya had a very modest but also a very strong and determined personality. He always spoke in a low voice and avoided, if possible, crowded and loud meetings. He often managed to find his own trails (not recommended in a conference bulletin), which always were less populated, more quiet, and therefore more suitable for thinking and enjoying nature. The same happened if there was a quiet hotel nearby (not recommended). Travel to remote places and hiking was Kolya's passion. Together with his hobby of fixing old huge American cars and vans, this made him one of the most knowledgeable around on the beauty of (often remote and hardly accessible) US landscapes. Again, everything that Kolya did he did very fundamentally. No wonder that the mechanics in Birmingham had the utmost respect for him. It is always the case that we read and write in reference letters that somebody is an expert in the field. With Kolya's passing away the mathematical community lost a great expert in billiards. He was the only one whom we could completely trust to judge that some very sophisticated proof on billiards dynamics is, in fact, correct. The memories of Prof. Chernov as an outstanding mathematician and a wonderful person will always be with those who were lucky enough to know Kolya. His scientific contributions will have a long lasting life. We lost a wonderful colleague and a dear dear friend.

Leonid Bunimovich

## My personal recollections on Kolya Chernov

by PÉTER BÁLINT (Budapest)

My first memories of meeting Kolya Chernov date back to the summer of 1998. This was the first time that I, as a graduate student, went abroad to a scientific conference, and I was excited to see the great experts, and Kolya Chernov in particular. I had spent quite some time studying his papers by then, which were fundamental for my work, and I was really amazed by the depth of his ideas, his technical skills, and the clarity of the exposition. This was a large conference with many participants, and it took me some time to realize that the thin and silent person with a kind smile on his face is actually the famous Chernov. He would always pay attention but only made comments at talks when it was really appropriate, and when not talking to someone, he was always deep in studying papers or working on his own arguments. But my impression enriched when we started to talk; Kolya was very much approachable from the first moment on. His explanations about the deeper connections beyond the recent developments of the field were truly enlightening for me. Besides mathematics, he also told me about himself, his family and his personal interests, and soon I realized what a colorful person he is. Our discussions also concerned some open problems, and he was very generous sharing with me his ideas, and answering patiently all my naive questions.

When I got home after the conference, with my advisor, Domokos Szász, and fellow graduate student Imre Péter Tóth, we started to work on a really difficult topic, statistical properties of multidimensional dispersing billiards. By the way, it is worth noting that despite much effort and good progress since then, this problem has not been solved to complete satisfaction even today. The three of us could solve certain aspects of the problem, but were struggling with some technicalities concerning the description of unstable manifolds. At this point we asked Kolya if he was interested in joining us, sending him all we had, in particular, some ideas and arguments that seemed appropriate but were lacking the fortunate formulation. It was in the middle of the semester, so Kolya did not reply immediately, but in approx. a month he sent us a document that contained a complete discussion of unstable manifolds, with all the technical properties properly formulated and the arguments worked out to the last detail. We were truly amazed... So the four of us started to collaborate on this topic. To summarize in a nutshell how this work progressed, we made some unexpected discoveries concerning the singularities of the system. This, even though it prevented reaching our original goals, concluded in two substantial papers, which contributed to a large extent to my PhD work.

For a couple of years I contacted Kolya Chernov only occasionally, until the winter of 2004 when I was about to apply for some postdoctoral grants, so I asked him for a reference letter. In his reply he wrote that he had just wanted to contact me if I had seen his recent manuscript with Dima Dolgopyat available from his webpage, and whether I was interested in working on related problems. This manuscript was the first version of

“Brownian Brownian motion I” that later grew to almost 200 pages long. I immediately understood that this work is revolutionary and got very excited, although it was clear that they were far ahead of me. Chernov and Dolgopyat wrote many more papers in the subsequent years, and in my opinion their collaboration is arguably the most influential at the interface of hyperbolic dynamics and statistical physics. The pair of them provided an amazing combination of innovative ideas, conceptual insight and technical strength, and I feel very fortunate that I could observe them in action from close proximity. They offered me several times to be involved, but it happened only very rarely that I could make a substantial contribution.



Once when I visited Kolya at UAB he was working on his book joint with Roberto Markarian entitled “Chaotic billiards.” He showed me some sections and asked about my opinion, he was really concerned that this book is properly written. Now the Chernov-Markarian book – the Book – is a basic everyday reference, a must for anyone who wants to work in this field. I think that popularizing the subject of billiards and making it accessible to a wide audience was in a sense a mission for Kolya Chernov. Hyperbolic billiards are technically difficult, their study requires a geometric approach and simultaneously involved analytic reasoning along with lengthy computations. For several decades it was only a small group of people who worked on this field, but now it has changed and billiards are considered as a more or less standard topic in dynamics. This is, to a large extent, Kolya Chernov’s merit. It is very fortunate that the strongest expert in the billiards community was so patient in

his explanations, and so determined when writing up the arguments. He discussed this involved topic in a crystal clear manner, presenting everything up to the smallest detail in all his papers, and especially in “the Book”.

Even though it might not have been so apparent because of his silent and modest character, there is no doubt that Kolya Chernov has been one of the leading figures in the theory of dynamical systems. His passing away at such a young age is a serious loss that is very hard to accept. For many of us, he was not only a great mathematician but also a wonderful person and most importantly, a close friend. To get some more impression what he meant for those who knew him, it is worth taking a look at the memorial website <http://nikolai-chernov.last-memories.com>, created and maintained by his former student, Hongkun Zhang. To me, Kolya Chernov’s attitude towards work and life will always be a role model and a source of inspiration, and his kind and wise personality remains in the middle of my heart.



## Quantum Mathematical Physics: A Bridge between Mathematics and Physics

September 29 - October 2, 2014, in Regensburg, Germany

In 2003, Jürgen Tolksdorf founded a series of conferences on quantum field theory and gravity: conceptual and mathematical advances in the search for a unified framework. In the tradition of the meetings in Blaubeuren/South Germany (2003 and 2005), Leipzig (2007), Regensburg (2010), it was the goal of the Regensburg conference 2014 to bring together about hundred physicists and mathematicians working in quantum field theory and general relativity and to encourage scientific discussions on fundamental and conceptual issues. Given by invited speakers, the selected talks introduced different directions of research, in a way as non-technical and easily accessible as possible. The conference was also intended for young researchers on the graduate and post-graduate level. The following talks were given:

- Christian Bär (Potsdam); “Characteristic Cauchy problem for wave equations on manifolds”
- Claudio Dappaggi (Pavia, Italy); “On the construction of Hadamard states from null infinity”
- Dirk Deckert (Davis, USA); “Time evolution of the Dirac sea subject to an external field for initial data on Cauchy surfaces”
- Michael Dütsch (Göttingen) “Massive vector bosons: Is the geometrical interpretation as a spontaneously broken gauge theory possible at all scales?”
- Bertfried Fauser (Cookeville, USA) “From software testing to differential geometry - computer science methods applied to physics”
- Chris Fewster (York, England); “The general theory of quantum field theory on curved spacetimes”
- Felix Finster (Regensburg); “Causal fermion systems as an approach to quantum theory”

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- Christian Fleischhack (Paderborn); “Loop quantization versus symmetry reduction”
  - Jose Gracia-Bondia (Zaragoza, Spain) “On the recursive evaluation of Feynman amplitudes in position space; differential versus Epstein-Glaser methods”
  - Michael Gransee (Leipzig); “Local thermal equilibrium states in quantum field theory and a generalization of the Kubo-Martin-Schwinger (KMS) condition”
  - Harald Grosse (Wien); “A nontrivial four-dimensional quantum field theory (in non-commutative geometry), Part I”
  - Christian Hainzl (Tübingen); “Mathematical aspects of many-particle quantum systems”
  - Stephan Hollands (Leipzig); “Dynamical versus thermodynamical (in)stability of black objects in gravity”,
  - Enno Kessler (Leipzig); “A super-conformal action functional for super Riemann surfaces”
  - Michael Kiessling (Rutgers University, USA); “The Dirac equation and the Kerr-Newman spacetime”
  - Gandalf Lechner (Leipzig); “The structure of the field algebra in non-commutative quantum field theory and uniqueness of its Kubo-Martin-Schwinger (KMS) thermal equilibrium state”
  - Martin Reuter (Mainz); “Quantum gravity, background independence and asymptotic safety”
  - Israel Sigal (Toronto, Canada) “Asymptotic completeness of Rayleigh scattering”
  - Christoph Stefan (Potsdam); “Non-commutative geometry in the era of the Large Hadron Collider at CERN”
  - Alexander Strohmaier (Loughborough, England); “The quantization of the electromagnetic field and its relation to spectral geometry and topology”,
  - Stefan Teufel (Tübingen) “Dimensional reduction for the Laplacian”
  - Roderich Tumulka (Rutgers University, USA); “Novel type of Hamiltonians without ultraviolet divergence for quantum field theories”,
  - Rainer Verch (Leipzig); “Linear hyperbolic partial differential equations with non-commutative time”
  - Stefan Waldmann (Würzburg); “Recent development in deformation quantization”

- Raimar Wulkenhaar (Münster); “A nontrivial four-dimensional quantum field theory (in non-commutative geometry), Part II.”

This program was complemented by the following two brilliant evening lectures:

- Siegfried Bethke (Max Planck Institute for Physics, Werner Heisenberg, Munich and CERN/Geneva); “Habemus Higgs - the story of discovering the Higgs boson,”
- Gerhard Börner (Max Planck Institute for Astrophysics, Garching); “Cosmic puzzles - dark matter and dark energy.”

The conference was financially supported by the German National Academy of Sciences Leopoldina, the Hans-Vielberth Foundation Regensburg, the Kepler Center Regensburg, IAMP, German National Science Foundation (DFG), Max Planck Institute for Mathematics in the Sciences, Leipzig. The organizers are very grateful to IAMP for the financial support given to young visitors of the conference.

Felix Finster, Jürgen Tolksdorf, and Eberhard Zeidler

## News from the IAMP Executive Committee

### New individual members

IAMP welcomes the following new members

1. ANDREAS ANDERSSON, School of Mathematics and Applied Atatistics, Wollongong University, Australia
2. DR. BENJAMIN BAHR, II. Institut für Theoretische Physik, Universität Hamburg, Germany
3. DR. DAVID ALEXANDRE ELLWOOD, Mathematics Department, Harvard University, Cambridge, MA, USA
4. DR. EDWARD IFIDON, Mathematics Department, University of Benin, Benin City, Nigeria
5. PROF. KEIICHI KATO, Mathematics Department, Tokyo University of Science, Japan
6. DR. SANGEET KUMAR, Mathematics Department, Chitkara University, Punjab, Rajpura, India
7. DR. SAEID MOLLADAVOUDI, Institute for Mathematical Research, University Putra Malaysia, Serdang, Malaysia
8. DR. ALEXANDER SCHENKEL, Department of Mathematics, Heriot-Watt University, Edinburgh, United Kingdom
9. DR. GIUSEPPE GENOVESE, Mathematisches Institut, Universität Zürich, Switzerland
10. PROF. HANNO GOTTSCHALK, Fachbereich C – Fachgruppe für Mathematik und Informatik, Bergische Universität Wuppertal, Wuppertal, Germany
11. DR. FABIAN PORTMANN, Department of Mathematical Sciences, University of Copenhagen, Denmark
12. DR. YOH TANIMOTO, Graduate School of Mathematical Sciences, The University of Tokyo, Japan
13. PROF. CRISTIAN GIARDINÀ, Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Modena, Italy
14. DR. KO SANDERS, Institut für Theoretische Physik, Universität Leipzig, Germany

## Recent conference announcements

### School on Current Topics in Mathematical Physics

August 3-8, 2015, Federico Santa María Technical University, Viña del Mar, Chile

organized by

Christian Hainzl, Mathieu Lewin, Robert Seiringer, Edgardo Stockmeyer, Rafael Tiedra

Web page <http://www.fis.puc.cl/~smp2015/frontpage.html>

This school is partially funded by the IAMP.

## Open positions

### Assistant Professor of Mathematics (Mathematical Physics) at ETH Zurich

The Department of Mathematics ([www.math.ethz.ch](http://www.math.ethz.ch)) at ETH Zurich invites applications for an assistant professor position (non-tenure track) in mathematical physics. Candidates should hold a PhD or equivalent in mathematics or physics, and should have demonstrated the ability to carry out independent research work. Willingness to teach at all university levels and to participate in collaborative work within mathematics and physics is expected. The new professor will be expected to teach undergraduate (German or English) and graduate courses (English) for students of mathematics, physics, and other natural sciences and engineering. He or she will be part of the National Competence Center in Research SwissMAP ([www.NCCR-SwissMAP.ch](http://www.NCCR-SwissMAP.ch)).

This assistant professorship has been established to promote the careers of younger scientists. The initial appointment is for four years with the possibility of renewal for an additional two-year period.

Please apply online at [www.facultyaffairs.ethz.ch](http://www.facultyaffairs.ethz.ch)

Applications should include a curriculum vitae, a list of publications, and a statement of your future research and teaching interests. The letter of application should be addressed to the President of ETH Zurich.

The closing date for applications is 28 February 2015.

### Postdoctoral position in mathematical quantum field theory at Harvard

A postdoctoral position is available with Arthur Jaffe at Harvard University in mathematical quantum field theory. The starting date is after July 1, 2015, with possible annual renewal until June 30, 2018. Apply by sending a letter, with a CV and list of publications to Mrs. Barbara Drauschke at [drauschke@fas.harvard.edu](mailto:drauschke@fas.harvard.edu). The applicant should also ask for two or three letters of recommendation to be sent by the recommenders directly to Mrs. Drauschke. Questions can be addressed to [Arthur\\_Jaffe@harvard.edu](mailto:Arthur_Jaffe@harvard.edu).

Harvard University is an equal opportunity employer, and all qualified applicants will receive consideration for employment without regard to race, color, religion, sex, national

origin, disability status, protected veteran status, or any other characteristic protected by law.

Applications will be accepted on a rolling basis until the position has been filled.

More job announcements are on the job announcement page of the IAMP

[http://www.iamp.org/page.php?page=page\\_positions](http://www.iamp.org/page.php?page=page_positions)

which gets updated whenever new announcements come in.

**Benjamin Schlein** (IAMP Secretary)

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