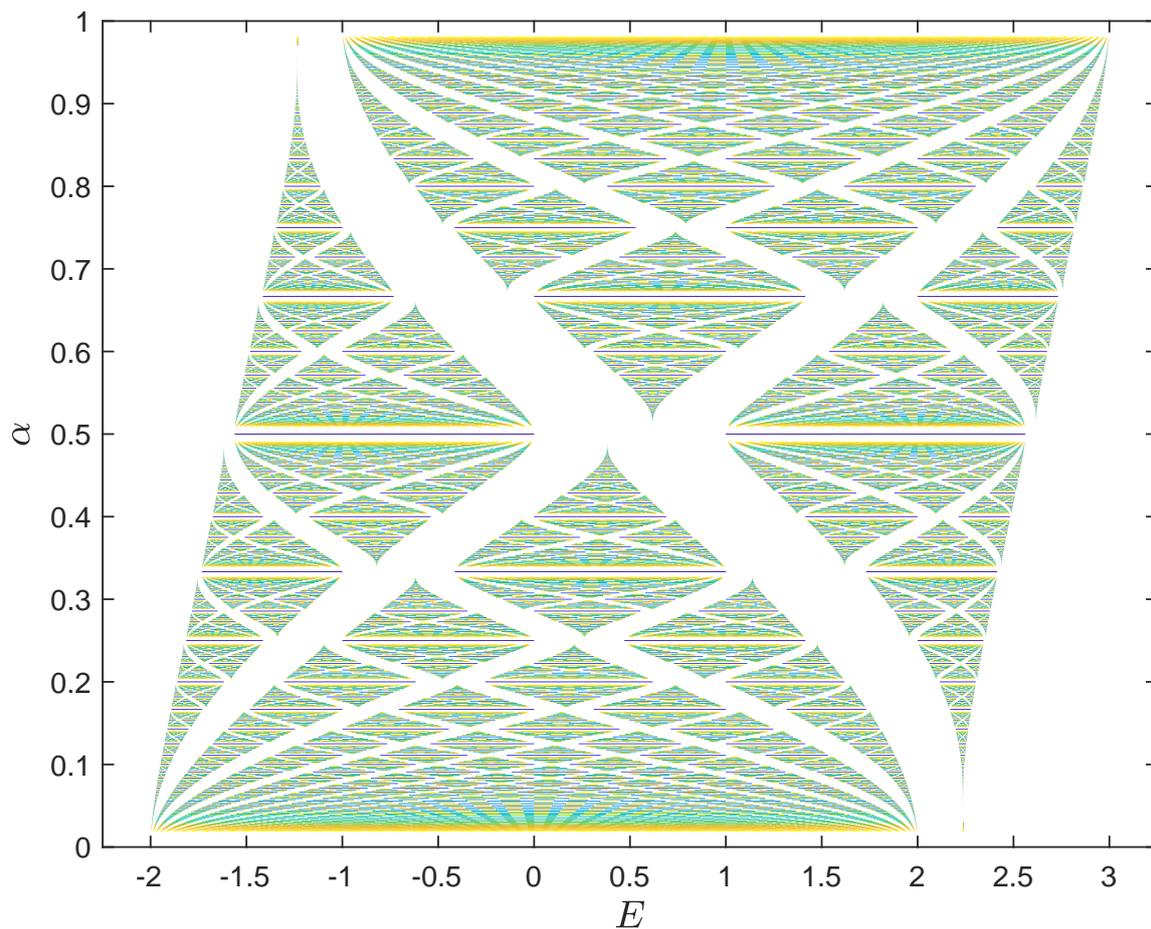


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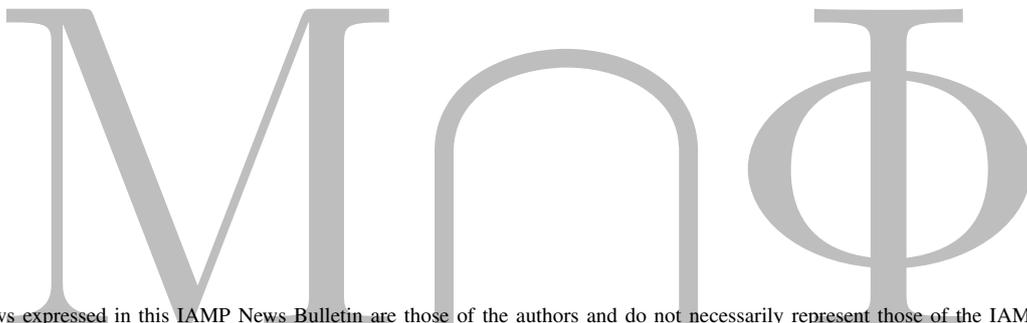
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Cover picture: A Sturmian analogue of Hofstadter's Butterfly, courtesy of Mark Embree



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Schrödinger operators with thin spectra

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The determination of the spectrum of a Schrödinger operator is a fundamental problem in mathematical quantum mechanics. We discuss a series of results showing that Schrödinger operators can exhibit spectra that are remarkably thin in the sense of Lebesgue measure and fractal dimensions. We begin with a brief discussion of results in the periodic theory, and then move to a discussion of aperiodic models with thin spectra.

1 Introduction

The purpose of this text is to discuss some recent work on Schrödinger operators with thin spectra. We will begin with a list of motivating questions and then focus on one of them, perhaps the most fundamental among them, and progressively ask more and more detailed refinements of this question, each of them prompted by a discussion of the previous one. This progression will lead us to the current state of affairs, where Schrödinger operators with surprisingly thin spectra have been identified, likely somewhat at odds with the expectations one might have had based on intuition or classical paradigms.

This note is not meant to be a comprehensive survey, and the questions we ask and the results we present will be selected according to the goal outlined in the previous paragraph: motivating and explaining the recent results on Schrödinger operators with thin spectra. For additional references and a discussion of the historical context we refer the reader to the original articles. Throughout the text, where appropriate, we point the reader towards suitable survey articles and textbooks for additional reading.

2 Schrödinger operators and things we might want to know about them

Schrödinger operators are central to the mathematical formulation of quantum mechanics; indeed their associated unitary group generates the time evolution of a quantum mechanical system. We will focus on the simplest case of a single

(quantum) particle exposed to a potential V and ignore all physical constants. Doing so we arrive at operators of the form $H = H_V$,

$$H = -\Delta + V \tag{2.1}$$

in

$$\mathcal{H} = L^2(\mathbb{R}^d),$$

where Δ is the standard *Laplacian*,

$$[\Delta\psi](x) = \sum_{j=1}^d \frac{\partial^2 \psi}{\partial x_j^2}(x), \tag{2.2}$$

and V acts by multiplication with the *potential* $V : \mathbb{R}^d \rightarrow \mathbb{R}$ (denoted by the same symbol, as is customary),

$$[V\psi](x) = V(x)\psi(x). \tag{2.3}$$

We will refer to d as the *dimension*. Its value is of crucial importance; there is a huge difference between the cases $d = 1$ and $d > 1$, both in terms of the results one should expect to hold and the methods that exist to establish these results. Even in the higher-dimensional case $d > 1$, the value of d is sometimes of importance, and there may be further transitions in the expected or provable behavior as d is increased.

It is often beneficial to initially consider the *free case* where V vanishes identically, so that one deals with $-\Delta$, and to only later add on the potential, essentially as a perturbation of $-\Delta$.

Naturally one cannot define (2.2) on all of \mathcal{H} , at least if one wants that object to belong to \mathcal{H} as well. Thus, it is an important problem to identify a suitable *domain* of this operator. It should not be too large (so that $-\Delta$ does not map some elements of the proposed domain outside \mathcal{H}) and it should also not be too small (for reasons that become important later: we want to have a well defined time evolution e^{-itH}). This leads to the goal of choosing the domain such that the operator is self-adjoint. Once this has been accomplished for the free case, the same domain works for large classes of potentials V , and hence the two-step procedure works out neatly in these cases. We won't dwell on this particular issue further, referring the reader instead to [50], and will assume henceforth that self-adjointness will always have been addressed successfully

for the Schrödinger operators H we consider. In a similar vein, most theorems may be formulated for various classes such as continuous potentials, smooth potentials, uniformly locally square-integrable potentials, and could even be pushed beyond that to even weaker regularity classes; to keep the exposition free of technicalities, we will mostly suppress assumptions on the regularity of V , and we will not attempt to state theorems in their absolute most general form. The reader is welcome to suppose that all potentials are bounded and continuous unless they are explicitly said to be otherwise.

Having realized (2.1) as a self-adjoint operator in \mathcal{H} , we can infer that the spectrum $\sigma(H)$ must be real. We will comment on the importance of this set momentarily. Let us first note that several interesting decompositions of this set exist. The most basic decomposition is

$$\sigma(H) = \sigma_{\text{disc}}(H) \cup \sigma_{\text{ess}}(H),$$

where the *discrete spectrum* $\sigma_{\text{disc}}(H)$ consists of all eigenvalues of H that have finite multiplicity, and the *essential spectrum* is the complementary set. In particular, this is a disjoint union, that is, a partition of the spectrum. By definition, and the fact that $L^2(\mathbb{R}^d)$ is separable, the discrete spectrum is a countable set, and it is therefore irrelevant for the questions we will discuss here – whether the spectrum is “thin.” Another important aspect, which however is less directly related to our theme, is that the dependence of $\sigma_{\text{disc}}(H)$ on V is very sensitive, while the dependence of $\sigma_{\text{ess}}(H)$ on V is far more robust.

In any event, let us formulate the first fundamental issue:

Fundamental Issue 1. Study the essential spectrum $\mathcal{S} = \sigma_{\text{ess}}(H)$.

Another consequence of self-adjointness is the existence of spectral measures guaranteed by the spectral theorem. Specifically, if H is self-adjoint and $\psi \in \mathcal{H}$, then there is a finite Borel measure $\mu_{H,\psi}$ on \mathbb{R} (indeed, $\mu_{H,\psi}(\mathbb{R}) = \|\psi\|^2$) such that

$$\langle \psi, (H - z)^{-1} \psi \rangle = \int_{\mathbb{R}} \frac{d\mu_{H,\psi}(E)}{E - z}, \quad z \notin \sigma(H). \quad (2.4)$$

In fact, each of the measures $\mu_{H,\psi}$ is supported by the spectrum, and hence we could integrate in (2.4) over $\sigma(H)$ rather than all of \mathbb{R} . For this reason, the structure of the spectrum places restrictions on the spectral measures.

For example, it is quite standard to decompose a Borel measure μ on \mathbb{R} into its pure point part $\mu^{(\text{pp})}$ (which has a countable support), its singular continuous part $\mu^{(\text{sc})}$ (which gives no weight to countable sets and has a support of zero Lebesgue measure), and its absolutely continuous part $\mu^{(\text{ac})}$ (which gives no weight to sets of zero Lebesgue measure). Thus, our second fundamental issue is the following:

Fundamental Issue 2. Study the spectral measures $\mu_{H,\psi}$ and their Lebesgue decomposition.

As a specific example of how information regarding the first fundamental issue may give information regarding the second fundamental issue, imagine that H is such that \mathcal{S} can be shown to have zero Lebesgue measure – it then immediately follows that $\mu_{H,\psi}^{(\text{ac})}$ vanishes for every $\psi \in \mathcal{H}$.

Why would the latter statement be of interest? As alluded to above, the time evolution of interest is given by the unitary group generated by H . In other words, since the time-dependent Schrödinger equation is given by

$$i \frac{\partial}{\partial t} \psi = H\psi, \quad \psi(0) = \psi_0, \quad (2.5)$$

self-adjointness of H and the spectral theorem again ensure that we can write the solution as

$$\psi(t) = e^{-itH} \psi_0. \quad (2.6)$$

The third fundamental issue is therefore the following:

Fundamental Issue 3. Study the quantum evolution $e^{-itH} \psi_0$.

There are several results (the RAGE Theorem [14] being the most prominent) that link the second and third fundamental issue. The Lebesgue decomposition of $\mu_{H,\psi_0}^{(\text{ac})}$ (e.g., information about which of the three components are non-trivial) gives information about the behavior of $e^{-itH} \psi_0$ as $|t| \rightarrow \infty$. As a rather obvious instance of this connection, let us note that by (2.4) and (2.6), the so-called *return probability*

$$|\langle \psi_0, e^{-itH} \psi_0 \rangle|^2 = \left| \int_{\sigma(H)} e^{-itE} d\mu_{H,\psi_0}(E) \right|^2$$

will go to zero as $|t| \rightarrow \infty$ if $\mu_{H,\psi_0} = \mu_{H,\psi_0}^{(\text{ac})}$ by the Riemann-Lebesgue Lemma.

To summarize, while we are ultimately interested in addressing the third fundamental issue, the standard approach to doing so involves addressing the other two fundamental issues as well. In this sense, a study of the (essential) spectrum of H is the most basic of these tasks, and it is the one to which we will devote the remainder of this text.

3 The essential spectrum \mathcal{S} and things we might want to know about it

As explained in the previous section, we wish to study the essential spectrum $\mathcal{S} = \sigma_{\text{ess}}(H)$ of a Schrödinger operator H in \mathcal{H} . Inspired by the two-step procedure mentioned earlier, let us begin with the free case.

If V vanishes identically, one can see via the Fourier transform, which conjugates $-\Delta$ to multiplication with $|\xi|^2$ in $L^2(\mathbb{R}^d, d\xi)$, that $\sigma(H) = \sigma_{\text{ess}}(H) = \mathcal{S} = [0, \infty)$ since that is the (closure of the) range of the multiplication operator in question. Thus, in the free case, \mathcal{S} is connected, that is, it has no gaps.

Adding on a potential, one may first wonder whether this property is preserved. If the operator V is relatively compact with respect to $-\Delta$, which will hold if the function V decays at infinity in a suitable sense, then the essential spectrum is indeed preserved and continues to be $[0, \infty)$. If relative compactness fails, one cannot immediately deduce that the essential spectrum is not preserved, but one may confidently expect that this should at least be possible for some such perturbations of $-\Delta$. Thus, one should expect a positive answer to the following question:

Question 1. Can \mathcal{S} have gaps, that is, can it be disconnected?

We will describe a mechanism that leads to a positive answer in Section 4: in one dimension, it suffices to consider (non-constant) periodic potentials V . These potentials do not decay at infinity but they have a sufficiently simple structure that their essential spectra can be studied quite effectively.

Once it is understood that \mathcal{S} may have gaps, one may then ask if one can have arbitrarily many, or even infinitely many, gaps. Thus, we refine the previous question as follows:

Question 2. Can \mathcal{S} have infinitely many gaps/infinitely many connected components?

This question can be answered again in the framework of non-constant periodic potentials, but there is an additional wrinkle: the dimension d now matters a lot! When $d = 1$, it is quite easy to produce infinitely many gaps, while in the case $d > 1$ it is not only seemingly hard, it is even known to be impossible under relatively weak assumptions. See Section 5 for details.

Once infinitely many gaps are present, \mathcal{S} has infinitely many connected components. Must they be intervals or is it ever possible that the gaps of \mathcal{S} are dense, either in all of \mathbb{R} or some portion of it? In other words, can \mathcal{S} (or a portion thereof) ever be a (generalized) Cantor set in the sense that it has empty interior?¹ This is the next question we may ask on our quest towards possible thinness phenomena:

Question 3. Can the gaps of \mathcal{S} be dense? In other words, is it possible that \mathcal{S} has empty interior?

The occurrence of Cantor spectra has been a hot topic since the 1980s. To exhibit this phenomenon one has to leave the class of periodic potentials, since Cantor structures are easily seen to be impossible in the periodic case. But simply taking the closure of the class of periodic potentials in a suitable way does the trick! In other words, potentials that are *almost-periodic* are good candidates when looking for Schrödinger operators with Cantor spectra. Much of the existing theory requires $d = 1$. This is not surprising as we mentioned above that gaps at high energies are atypical for periodic potentials when $d > 1$ and approximation of almost-periodic potentials by periodic potentials is a natural tool that should also inform one's expectations about the limit spectra that arise. However, very recently it has been shown how Cantor spectra can be shown to arise even in the higher-dimensional setting. We will discuss all of these points in Section 6.

For nowhere dense sets, which are quite thin in a topological sense, one can explore further thinness features by considering measure-theoretic properties. Starting with Lebesgue measure, one may ask the following:

Question 4. Can the Lebesgue measure of \mathcal{S} vanish?

¹Since all operators that we consider are unbounded, the corresponding spectra are also unbounded and hence never compact. To avoid constantly saying “generalized Cantor set”, we will abuse terminology somewhat and say a Cantor set is a closed set with empty interior and no isolated points.

While the Cantor set just about anyone is first exposed to is the middle third Cantor set, which is easily seen to have zero Lebesgue measure, further explorations then show that there are also “fat Cantor sets” of positive Lebesgue measure. In the context of Schrödinger spectra the earliest examples turned out to have positive Lebesgue measure, but further, more quantitative, work then was able to give a positive answer to the previous question; see Section 7 for details.

Once the Lebesgue measure of \mathcal{S} is shown to be zero, this naturally prompts us to consider fractal dimensions. Specifically, since Lebesgue measure on \mathbb{R} is just the α -dimensional Hausdorff measure for $\alpha = 1$, it is natural to lower the value of α until \mathcal{S} no longer has zero weight with respect to h^α . This transition value is called the *Hausdorff dimension* of \mathcal{S} . There are other ways of studying whether \mathcal{S} has fractional dimension in a suitable sense, for example by considering the (lower or upper) box counting (or Minkowski) dimension. If a fractional value occurs for one of these dimensions, it is then of interest to look at the others and see whether they yield the same value, or whether there are essential spectra for which the different fractal dimensions indeed differ. Since the spectrum is a subset of \mathbb{R} , all these fractal dimensions of \mathcal{S} will take values in $[0, 1]$. Once one confirms that values below 1 are possible, one naturally asks whether dimensions all the way down to, and including, zero are possible:

Question 5. Can \mathcal{S} be zero-dimensional?

Section 8 explains that this is indeed possible, and this will demonstrate that the existing results have pushed the possible thinness of essential spectra of Schrödinger operators to the extreme, in the sense of topology, measure theory, and fractal geometry. But there is an important additional aspect in pursuing this quest for extreme results. Only once zero-dimensionality has been established in the case $d = 1$, can one then address Questions 3–5 in the case $d > 1$. See Section 8 for this discussion.

After all of these results, one may wonder whether there is any lower bound that one can impose on the essential spectrum.

Question 6. Is there any sense in which the size of the set \mathcal{S} can be bounded from below?

We will take this up in Section 9. Let us note that if $V(x)$ is unbounded, then it is possible for H to have compact resolvent and hence for $\sigma(H)$ to con-

sist of isolated eigenvalues of finite multiplicity which only accumulate at $+\infty$. However, in this case $\mathcal{S} = \emptyset$, so we do not view this as an interesting counterexample. One may observe this explicitly for (say) the quantum harmonic oscillator ($V(x) = x^2$), which can be explicitly diagonalized in terms of Hermite functions; compare [49, p. 142] or [57, Section 6.4]. Consequently, it is mainly of interest to exhibit the desired spectral results within (say) the class of bounded potentials, and we shall restrict attention to that case in this note.

For this note, we focus primarily on operators as in (2.1), but it will sometimes be useful to discuss the related discrete Schrödinger operators given by choosing a bounded potential $v : \mathbb{Z}^d \rightarrow \mathbb{R}$ and defining $h = h_v = -\Delta_{\mathbb{Z}^d} + v$ in $\ell^2(\mathbb{Z}^d)$, where

$$[\Delta_{\mathbb{Z}^d}\psi](n) = \sum_{\|m-n\|_1=1} \psi(m),$$

and $[v\psi](n) = v(n)\psi(n)$ for $n \in \mathbb{Z}^d$.

4 Can \mathcal{S} have gaps?

As discussed, if $V \equiv 0$, the spectrum of H is the half-line $[0, \infty)$, and if V enjoys suitable decay at $\pm\infty$, the essential spectrum remains undisturbed. Thus, it becomes necessary to work with nonzero potentials that do not decay. Of course, if $V(x) \equiv \lambda_0$ a constant, then $\sigma(H) = [\lambda_0, \infty)$, so it is necessary to work with nonconstant potentials as well. The simplest potentials to work with are those that are periodic; we say $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is *periodic* if there is a basis $\{x_1, \dots, x_d\}$ of \mathbb{R}^d such that $V(x + x_j) \equiv V(x)$ for every $1 \leq j \leq d$.

It is an amazing result from the inverse theory that, in one space dimension, *every* non-constant periodic potential has a gap in its spectrum.

Theorem 4.1. *If $V : \mathbb{R} \rightarrow \mathbb{R}$ is periodic and nonconstant, \mathcal{S} has a gap.*

This result is originally due to Borg [8], with a short self-contained proof due to Hochstadt [29]; see also the historical remarks and references in [29] for more.

Let us say a little bit more about the determination of the spectrum when V is periodic. We emphasize that the approach we discuss here is strictly one-dimensional; we will discuss the higher-dimensional setting later. For simplicity, take V to have period one, and let u_1, u_2 solve the initial-value problem

$-u'' + Vu = Eu$ subject to initial conditions

$$\begin{bmatrix} u_1'(0) & u_2'(0) \\ u_1(0) & u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the *monodromy matrix*

$$\Phi(E) = \begin{bmatrix} u_1'(1) & u_2'(1) \\ u_1(1) & u_2(1) \end{bmatrix}$$

and its trace $D(E) := \text{Tr}[\Phi(E)]$ tell essentially the whole story about general solutions to

$$-u'' + Vu = Eu, \tag{4.1}$$

in that one has

$$\begin{bmatrix} u'(n) \\ u(n) \end{bmatrix} = \Phi(E)^n \begin{bmatrix} u'(0) \\ u(0) \end{bmatrix}$$

for all $n \in \mathbb{Z}$. Thus, asymptotic characteristics of $u(x)$ are entirely dictated by behavior of large powers of $\Phi(E)$. In particular, one has the following possibilities:

- $D(E) \in (-2, 2)$ or $\Phi(E) = \pm I$. Every solution of (4.1) is uniformly bounded, so E is a generalized eigenvalue of H_V , and hence E belongs to the spectrum of H_V by Sch'nol's Theorem [52, 55].
- $D(E) = \pm 2$ and $\Phi(E) \neq \pm I$. Solutions of (4.1) are linearly bounded, and E again belongs to the spectrum of H_V by Sch'nol's theorem.
- $D(E) \notin [-2, 2]$. There are linearly independent solutions u_{\pm} of (4.1) such that u_{\pm} decays exponentially at $\pm\infty$ and grows exponentially at $\mp\infty$, and $E \notin \sigma(H)$. One can appeal to Sch'nol again or use u_{\pm} to explicitly construct the resolvent of H at E .

One can say a bit more about D . Since the spectrum of H_V must be real, any value of E with $D(E) \in [-2, 2]$ must be real. In particular, D has all real roots. Moreover, if $D'(E) = 0$ for some $E \in \mathbb{R}$, one must have $D(E) \notin (-2, 2)$, for, if not, then one could produce nearby nonreal E with $D(E) \in$

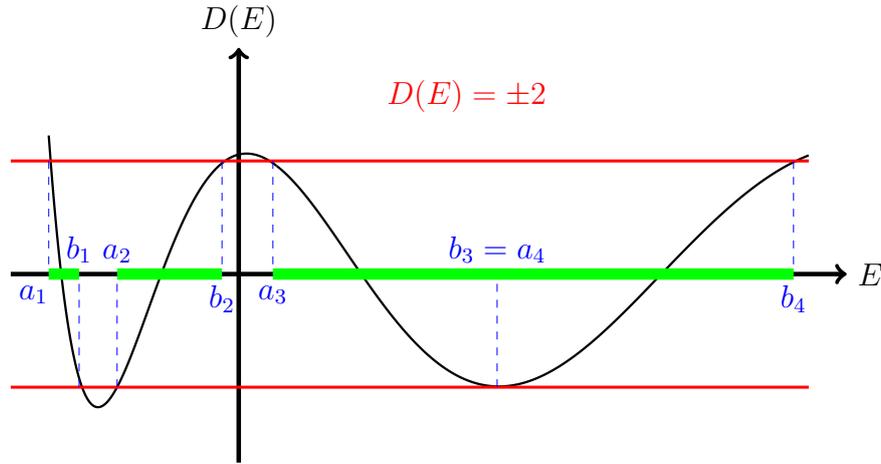


Figure 4.1: Determination of \mathcal{S} from D .

$(-2, 2)$, contradicting self-adjointness. Thus, \mathcal{S} is a union of nondegenerate closed bands,

$$\mathcal{S} = \bigcup_{k=1}^{\infty} [a_k, b_k],$$

where each band $B_k = [a_k, b_k]$ is obtained as the closure of a connected component of the preimage of $(-2, 2)$ under D . See Figure 4.1 for a visualization of $D(E)$ and how it determines \mathcal{S} .

In view of this picture of D , one sees that a gap may only close at energy E_0 if $D(E)$ is tangent to one of the horizontal lines at height ± 2 at E_0 . Consequently, it seems like the presence of a closed gap is an unstable phenomenon, and one might expect that a “typical” potential breaks all of these gaps open. We discuss this further in the next section.

Of course, the approach outlined above does not work for $d \geq 2$, but a closer inspection suggests the correct generalization. Namely, given $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, we may have $D(E) = 2 \cos(2\pi\theta) \in [-2, 2]$ if and only if there is a solution u to (4.1) such that $u(x+1) = e^{2\pi i\theta} u(x)$, and it is this notion which makes perfect sense in higher dimensions. Namely, if V is periodic and $\{x_1, \dots, x_d\}$ is a basis of \mathbb{R}^d for which $V(x) \equiv V(x + x_j)$ for every j , then one seeks solutions of

$$-\Delta u + Vu = Eu \tag{4.2}$$

such that

$$u(x + x_j) = e^{2\pi i\theta_j} u(x) \tag{4.3}$$

for some $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$. For every θ , there is a countable set $E_1(\theta) \leq E_2(\theta) \leq \dots$ such that (4.2) enjoys a nontrivial solution for $E = E_j(\theta)$ satisfying (4.3) (we enumerate the E_j 's according to multiplicity, which is why we write “ \leq ”). Then the k th band is given by $B_k = \{E_k(\theta) : \theta \in \mathbb{T}^d\}$, and one has

$$\mathcal{S} = \bigcup B_k.$$

Each E_k can be shown to be non-constant, so each B_k is a non-degenerate closed interval. The primary difference between the lower- and higher-dimensional cases is that the bands could only touch in $d = 1$, but when $d \geq 2$, the bands may *overlap* (i.e. one may have $b_k > a_{k+1}$). This will be responsible for some of the differences one sees in the higher-dimensional setting, including the added difficulty of constructing examples having Cantor spectrum.

5 Can \mathcal{S} have infinitely many gaps?

The answer to this question (in the periodic setting) depends entirely on the dimension. If $d = 1$ and, for example, $V(x) = \cos x$, then it is known that \mathcal{S} is a disjoint union of infinitely many compact intervals [51, pp. 298–299]. In other words, the spectrum has infinitely many gaps at arbitrarily high energies. This topological structure is known to be generic² among all one-dimensional periodic potentials. For the sake of notation, let $C_{\text{per}}^r(\mathbb{R})$, $r \in \mathbb{Z}_+ \cup \{\infty\}$ denote those $V \in C^r(\mathbb{R})$ such that $V(x+1) \equiv V(x)$, equipped with the uniform topology for $r = 0$ and the usual Fréchet topology when $r > 0$.

Theorem 5.1 (Simon, 1976 [54]). *For generic $V \in C_{\text{per}}^r(\mathbb{R})$, $b_k < a_{k+1}$ for all k . In particular, \mathcal{S} has infinitely many open gaps.*

On the other hand, the picture in higher dimensions is starkly different. In general, the arguments become quite complex, but one can get a simple idea for how things change in the following simple scenario. First, consider a given $V \in C_{\text{per}}^0(\mathbb{R})$. Denoting the k th band and gap by $B_k = [a_k, b_k]$ and $G_k = (b_k, a_{k+1})$

²Here, and throughout this note, a property that may hold at some point in a topological space is said to be generic if the set on which it holds is a dense G_δ .

respectively, and using $|\cdot|$ to denote the length of an interval, we have:

$$\sum_k \left(\pi^2(2k-1) - |B_k| \right)^2 < \infty \quad (5.1)$$

$$\sum_k |G_k|^2 < \infty; \quad (5.2)$$

see [24, 25]. (In fact, the summands of (5.1) and (5.2) are very easily seen to be bounded by $4\|V\|_\infty^2$ by general perturbation theory, and this already suffices for the purposes that we shall undertake presently). Now, let us see what insights this yields in higher dimensions. A potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is called *separable* if there are potentials $W_j : \mathbb{R} \rightarrow \mathbb{R}$ for $1 \leq j \leq d$ so that

$$V(x) = \sum_{j=1}^d W_j(x_j).$$

Writing $\mathcal{S} = \sigma(H_V)$ and $\mathcal{S}_j = \sigma(H_{W_j})$, the general theory (see, for instance, [49, Chapter VIII]) shows that \mathcal{S} decomposes as a Minkowski sum:

$$\mathcal{S} = \sum_{j=1}^d \mathcal{S}_j = \left\{ \sum_{j=1}^d E_j : E_j \in \mathcal{S}_j \right\}. \quad (5.3)$$

Combining the observations (5.1), (5.2), and (5.3) shows the following: if $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous separable periodic potential, then $\sigma(H_V)$ has only finitely many gaps. On the basis of this calculation, the following result was quite long-expected:

Conjecture 5.2 (Bethe–Sommerfeld Conjecture). *Suppose $d \geq 2$. If $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is periodic and is such that H_V is self-adjoint, then $\sigma(H_V)$ has finitely many gaps.*

This result is quite difficult to prove in the non-separable case, and has only recently been concluded. An incomplete list of contributions includes [28, 32, 59, 60, 61, 66], with the current best available result in [43]. Versions for discrete Schrödinger operators may be found in [22, 34], with the most general discrete result in [26]. For more on the history and development of this problem (and the spectral theory of periodic operators writ large), we recommend Kuchment’s survey [35].

6 Can the gaps of S be dense?

From the previous discussion, the answer must be a resounding “no” if one restricts to the case of periodic potentials. Another scenario that was studied intensively at first was the case of random potentials. Under relatively mild assumptions, it is known that the spectrum of a random operator is the same for almost every realization and that this almost-sure spectrum is given by the closure of the union of the spectra of corresponding periodic realizations; compare [33]. In particular, if there are any gaps in the (almost-sure) spectrum, they do not accumulate. Thus, in the random and periodic cases, the associated operators (almost) always have spectra in which the gaps cannot be dense. Let us note in passing that the study of random operators is itself a deep field with a vast literature; we refer the reader to [1, 63] and references therein.

However, in the 1970s and 1980s, interest began to grow in aperiodic potentials with suitable recurrence properties, motivated by several developments in math and physics. First, the study of electrons subjected to external magnetic fields (which dates back to the early part of the 20th century, compare [27, 40, 46]) led to interest in quasi-periodic models. Famously, plots of the spectra corresponding to Harper’s model led to a picture now known as Hofstadter’s butterfly [30], one of the first fractals observed in physical scenarios. These models led to interest in the almost-Mathieu operator (AMO), which is a quasi-periodic discrete Schrödinger operator h_v with

$$v(n) = v_{\lambda, \alpha, \omega}(n) = 2\lambda \cos(2\pi(n\alpha + \omega)) \quad (6.1)$$

for $n \in \mathbb{Z}$, where $\lambda, \alpha, \omega \in \mathbb{R}$ are called the *coupling constant*, *frequency*, and *phase*, respectively. We recommend [15, 41] for more about such quasi-periodic operators, their history, and connections to physics. Of course, if α is rational or $\lambda = 0$, the resulting v is periodic and hence can be handled by a formalism analogous to that described in Section 4, so the most interesting case is that in which α is irrational and $\lambda \neq 0$. The second major physical development that spurred mathematical study of aperiodic operators was the discovery of quasicrystals by Shechtman *et al.* in the early 1980s [53]. We will not give a comprehensive account of mathematical quasicrystals, instead referring the reader to [6] and references therein for background. The central operator-theoretic model of a quasicrystal is the *Fibonacci Hamiltonian*, a dis-

crete Schrödinger operator with potential given by

$$v(n) = v_{\lambda,\omega}(n) = \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1), \quad (6.2)$$

where $\lambda > 0$ and $\omega \in \mathbb{R}$ are parameters, again called the coupling constant and the phase. Here, α is fixed to be $\frac{1}{2}(\sqrt{5} - 1)$, the inverse of the golden mean (however, one can also choose other irrational values of α , leading to *Sturmian* potentials).

Let us briefly comment on the similarities and differences between (6.1) and (6.2). Both are aperiodic, but both are also “not too far” from being periodic, however in vastly different ways. As an almost-periodic potential, (6.1) exhibits ε -almost periods, that is, nontrivial translations t such that $\|v - v(\cdot - t)\|_\infty < \varepsilon$. The Fibonacci potential (6.2) is not almost-periodic, indeed has no nontrivial ε -almost-period for any $\varepsilon \leq \lambda$. Instead, there are nontrivial translations t_1, t_2, \dots and $R_j \rightarrow \infty$ such that v and $v(\cdot - t_j)$ agree on $[-R_j, R_j]$. To summarize, (6.1) enjoys imprecise repetitions at a global scale, while (6.2) enjoys exact local repetitions. These differences inform the different properties of these operators, as well as the different tools and techniques that have been brought to bear in their study.

To study an operator with a potential such as (6.1) (or a suitable analogue in $L^2(\mathbb{R})$), a natural tactic is to approximate α by rationals, study the resulting periodic operators, and try to prove suitable approximation results. The problem is that such approximations are inherently local. In particular, these approximations will be close to the original aperiodic potential on a finite window, and will not at all be good approximations outside that window. To put this into the language of operator theory, the resulting periodic approximants approximate the aperiodic operator strongly but not in norm, so one is naturally interested in what operators enjoy a periodic approximation in the norm topology. To wit: a potential $V \in C(\mathbb{R}^d)$ is called (uniformly) *limit-periodic* if it may be uniformly approximated by periodic elements of $C(\mathbb{R}^d)$. For the case of limit-periodic operators, one might expect their spectral properties to correspond closely to those in the periodic case, but we will see that they can diverge quite far. Let $\text{LP}(\mathbb{R}^d)$ denote the set of limit-periodic $V \in C(\mathbb{R}^d)$, which is a complete metric space in the uniform metric (note however that it is not closed under sums and hence is not a Banach space). The first result on Cantor spectrum in the limit-periodic case is due to Moser:

Theorem 6.1 (Moser, 1981 [42]). *There exist $V \in \text{LP}(\mathbb{R})$ such that \mathcal{S} is a Cantor set.*

Very shortly after Moser's work, Avron–Simon [5] showed that Cantor spectrum is generic in $\text{LP}(\mathbb{R})$. Let us also mention related work of Chulaevskii [12, 13] and Pastur–Tkachenko [44, 45]. For more about limit-periodic operators in general, see the survey [16] and references therein.

Once one understands how the process of periodic approximation works, one is no longer surprised by theorems showing that Cantor spectrum is generic in the limit-periodic setting. There are a few basic heuristics: First, one can show from the definitions that the spectrum never has isolated points. Next, one can show that Cantor spectrum is a G_δ phenomenon, which is essentially an exercise in point-set topology and perturbation theory. Concretely, if I is an open interval, the set of V for which $I \cap \rho(H_V) \neq \emptyset$ is easily shown to be open. Thus, the intersection of all such sets of potentials as I ranges over intervals having rational endpoints is a G_δ and is rather clearly the set of potentials enjoying nowhere-dense spectrum. Finally, one needs to show that Cantor spectrum is dense in the space $\text{LP}(\mathbb{R})$. This is more technical, but the outline is the following. Start with a periodic potential, and successively produce periodic perturbations of higher and higher periods for which all gaps that can open do open (by using Theorem 5.1), and in such a way that gaps opened in previous steps do not close, taking care that the gaps also cannot close in the limit.

Let us observe that the outline suggested in the previous paragraph does *not* work for dimensions $d > 1$. Concretely, Theorem 5.1 only holds for $d = 1$, and fails spectacularly for $d > 1$, in view of the Bethe–Sommerfeld conjecture. In particular, the presence of Cantor spectrum in higher dimensions is necessarily more difficult to prove than in the 1D case. Nevertheless, we will see in Section 8 that suitable quantitative refinements of the 1D Cantor results can be used to generate Cantor spectrum in higher dimensions.

Since we mentioned the AMO, let us note that it was later shown to have Cantor spectrum for all $\lambda \neq 0$, all $\omega \in \mathbb{R}$, and all irrational α [3]. Similarly, Cantor spectrum has been established for the Fibonacci (as well as the more general Sturmian) case for all non-trivial parameter values [7, 64]. See [15, 41] for further comments on the history and earlier partial results.

7 Can the Lebesgue measure of \mathcal{S} vanish?

As soon as one has a Cantor spectrum, one is interested in the measure properties. In particular, even if the gaps are dense, the spectrum may still be a “fat” Cantor set and hence have positive measure (or even infinite measure, since \mathcal{S} is unbounded). Returning briefly to the discrete setting, the first two operators known to have zero-measure spectrum of physical interest were the critical AMO ((6.1) with $\lambda = 1$) and the Fibonacci Hamiltonian [64], with potential given by (6.2). On one hand, the zero-measure result for the AMO is very sensitive to the coupling constant, since the measure of the spectrum (for irrational α) is known to be given by $|4 - 4|\lambda||$ for every λ (see e.g. [4, 31, 36, 37]). In particular, as soon as one perturbs the coupling constant, the zero-measure property is destroyed, even though the Cantor property persists. By way of contrast, the Fibonacci Hamiltonian exhibits zero-measure Cantor spectrum for arbitrary $\lambda \neq 0$, and hence, the zero-measure phenomenon is more robust in this scenario.

This discussion of stability is significant from the point of view of the desired results for operators of the form $-\Delta + V$, which are unbounded. In particular, the high-energy region for H_V is analogous to the regime of small coupling constant in the discrete case. In particular, one should not expect a zero-measure result to hold for an analogue of the AMO in $L^2(\mathbb{R})$, so, in order to generate examples of the desired phenomenon for Schrödinger operators in $L^2(\mathbb{R})$, one should look in (say) the Fibonacci setting, where the desired phenomenon is robust in the small coupling region.

To spare the reader technicalities and notation, let us work in a setting that is simpler than the most general possible setup, but which still exhibits the behaviors that we want to discuss. Let u denote the 0-1 Fibonacci sequence, given as:

$$u(n) = u(n, \omega) = \chi_{[1-\alpha, 1)}(n\alpha + \omega \bmod 1)$$

with $\alpha = \frac{1}{2}(\sqrt{5}-1)$ as before. Choosing $f_0 \neq f_1$ in $L^2([0, 1))$, one can generate a Fibonacci Schrödinger operator by replacing each 0 in u by f_0 and each 1 by

f_1 and attaching a coupling constant³ λ ; more precisely, take:

$$V_{\omega,\lambda}(n) = \lambda \sum_{n \in \mathbb{Z}} f_{u(n,\omega)}(\cdot - n) \chi_{[n,n+1)}. \quad (7.1)$$

Theorem 7.1 (D.–F.–Gorodetski, 2014 [17]). *For every $f_0 \neq f_1$, every $\lambda \neq 0$, and every $\omega \in \mathbb{R}$, \mathcal{S} is a Cantor set of zero Lebesgue measure.*

The result of [17] is more general than this: one can replace α by an arbitrary irrational number; in fact, one can replace these aperiodic 0-1 sequences by any element of a minimal subshift satisfying Boshernitzan’s criterion for unique ergodicity [9, 10], subject to a suitable local recognizability criterion on the local potential pieces; see [17] for details. See also [39] for similar results that were proved independently. We specialize to the Fibonacci case because of the physical motivation and because our later results about dimension will require it.

Additionally, the zero-measure property also holds for the case of limit-periodic potentials, and remains stable in the coupling constant for generic potentials, first shown in the discrete case by Avila [2].

Theorem 7.2 (D.–F.–Lukic, 2017 [20]). *For generic $V \in \text{LP}(\mathbb{R})$, $\sigma(H_{\lambda V})$ is a zero-measure Cantor set for every $\lambda \neq 0$.*

The latter claim is somewhat surprising, since one knows that $\sigma(H_{\lambda V})$ converges to $\sigma(H_0) = [0, \infty)$ in the Hausdorff metric as $\lambda \rightarrow 0$. In particular, as $\lambda \downarrow 0$, the resolvent set of $H_{\lambda V}$ remains a dense, full-measure subset, even though the length of every gap goes to zero at least as fast as $2\lambda\|V\|_\infty$.

8 Can \mathcal{S} be zero-dimensional?

Drawing on ideas from Avila [2], one can refine the Cantor spectrum arguments in a quantitative way. The first step is to go from qualitative behavior to quantitative behavior in the periodic case. Concretely, we know (from Theorem 5.1) that generic periodic potentials open all possible gaps in dimension one. One of the key insights of Avila in the discrete setting was a mechanism that one can use to generate not only gaps, but to show that the spectrum of an operator of period p can be made exponentially small (i.e. $\lesssim e^{-cp}$ for a suitable

³Of course, one could simply absorb λ into the f_j , but it is of interest to fix f_j first and then to observe the scaling behavior as the coupling constant is varied.

constant $c > 0$). Altering the original construction of limit-periodic operators to generate periodic approximants with such measure estimates at each stage, one can control the fractal dimensions of the limiting spectrum. Of course, in the continuum case, it does not make sense to speak of measure bounds on the whole spectrum of a periodic operator (which necessarily has infinite measure, compare (5.1)), so some care is needed. Nevertheless, one can show

Theorem 8.1 (D.–F.–Lukic, 2017 [20], D.–F.–Gorodetski, 2019 [18]). *Within $LP(\mathbb{R})$, the set of V for which $\dim_{\mathbb{H}}(\sigma(H_V)) = 0$ is dense. In fact, for a dense set of $V \in LP(\mathbb{R})$, one has $\dim_{\mathbb{H}}(\sigma(H_{\lambda V})) = \dim_{\mathbb{B}}^-(\sigma(H_{\lambda V})) = 0$ for every $\lambda \neq 0$.*

The result about Hausdorff dimension is from [20], and the result about the lower box-counting dimension is from [18]. The strengthening from Hausdorff to box-counting dimension allows us to conclude answers to all of our questions about \mathcal{S} in higher dimensions.

Theorem 8.2 (D.–F.–Gorodetski, 2019 [18]). *There exist $V \in LP(\mathbb{R}^d)$ for any $d \geq 2$ such that \mathcal{S} is a Cantor set having zero Lebesgue measure, zero Hausdorff dimension, and zero lower box-counting dimension.*

The key fact that enables one to make the transition is the following observation. If A is a compact set, then

$$\dim_{\mathbb{B}}^- \left(\sum_{j=1}^d A \right) \leq d \dim_{\mathbb{B}}^-(A). \quad (8.1)$$

In particular, if A has zero lower box-counting dimension, so too does $\sum_{j=1}^d A$ for any d . Thus, Theorem 8.2 is proved by using Theorem 8.1 to construct a 1D potential W for which the associated spectrum has lower box-counting dimension zero, constructing a d -dimensional separable potential of the form

$$V(x) = \sum_{j=1}^d W(x_j),$$

and then invoking (8.1) and (5.3). We are sweeping a few niceties about fractal dimensions of sums of unbounded sets under the rug here, but they are not hard to deal with, see, e.g., the appendix to [18].

Let us note that one really does need the limit-periodic setting here. One can show that the critical AMO exhibits zero Hausdorff dimension for some frequencies [38]; however, we remind the reader that because these only hold for $\lambda = 1$, one cannot expect to be able to use them to deduce consequences for Schrödinger operators in $L^2(\mathbb{R})$. Moreover, one can prove dimensional results for the Fibonacci Hamiltonian, but one cannot zero out its Hausdorff dimension. In particular, one has the following for potentials $V_{\omega,\lambda}$ as in (7.1). Let us denote \mathcal{S}_λ the essential spectrum corresponding to $V_{\omega,\lambda}$ (the omission of ω from the notation reflects the fact that \mathcal{S} does not depend on ω , which can be seen from strong approximation arguments).

Theorem 8.3 (D.–F.–Gorodetski, 2014 [17], F.–Mei, 2018 [23]). *For any $f_0 \neq f_1$ in $L^2([0, 1])$ and any $\lambda > 0$, one has*

$$\dim_{\mathbb{H}}^{\text{loc}}(\mathcal{S}_\lambda, E) > 0$$

for every $E \in \mathcal{S}_\lambda$. In the small-coupling limit, one has:

$$\lim_{\lambda \rightarrow 0} \inf_{E \in \mathcal{S}_\lambda} \dim_{\mathbb{H}}^{\text{loc}}(\mathcal{S}_\lambda, E) = 1.$$

Moreover, for any $\lambda > 0$, in the high-energy limit, we have

$$\lim_{E_0 \rightarrow \infty} \inf_{E \in \mathcal{S}_\lambda \cap [E_0, \infty)} \dim_{\mathbb{H}}^{\text{loc}}(\mathcal{S}_\lambda, E) = 1.$$

In particular, the global Hausdorff dimension of \mathcal{S}_λ is always equal to 1.

The asymptotic statements for the dimension were proved for $f_0 \equiv 0$ and $f_1 \equiv 1$ in [17] and extended to general f_j in [23]. Let us now move to the higher dimensional setting. We fix $f_0 \equiv 0$, $f_1 \equiv \lambda$, and denote by \mathcal{S}_λ the spectrum of $H_{V_{\omega,\lambda}}$ as before. For $d \geq 2$, write

$$\mathcal{S}_\lambda^{(d)} = \sum_{j=1}^d \mathcal{S}_\lambda,$$

the d -fold sum (cf. (5.3)). In particular, $\mathcal{S}_\lambda^{(d)}$ is the spectrum of $H_{\lambda,\omega}^{(d)} := -\Delta + V_{\omega,\lambda}^{(d)}$, where

$$V_{\omega,\lambda}^{(d)}(x) = \sum_{j=1}^d V_{\omega,\lambda}(x_j), \quad x \in \mathbb{R}^d.$$

The dimensional results in Theorem 8.3 suggest (but do not imply!) that one has the following picture for the multidimensional operator $H_{\omega,\lambda}^{(d)}$: the spectrum undergoes a transition from Cantor-like at low energies to containing intervals and perhaps eventually a half-line for large energies. This was recently proved.

Theorem 8.4 (D.–F.–Gorodetski, 2020+ [19]). *Let $d \geq 2$. For any $\lambda > 0$, $\mathcal{S}_\lambda^{(d)}$ contains a half-line. There exists $C(d) > 0$ such that, for $\lambda \geq C(d)$, $\mathcal{S}_\lambda^{(d)}$ enjoys Cantor structure near the ground state, that is, there is an E_0 such that $(-\infty, E_0] \cap \mathcal{S}_\lambda^{(d)}$ is a nonempty Cantor set.*

9 When does it end?

With these increasingly fine results that show that the spectrum can be made arbitrarily small in the sense of Lebesgue measure and fractal dimensions, one wonders whether there is any hard limit at all to how far one can push this. In other words, is there some notion of the size of a set for which one can prove a *lower* bound on the size of the spectrum, valid for a large class of (bounded) potentials? Of course, as we already mentioned one does need to insert the word “bounded” here, since there are operators with unbounded potentials having compact resolvent, for which $\mathcal{S} = \emptyset$. It turns out that potential theory in the complex plane offers a partial answer. Let us emphasize that the results in this section only apply to one-dimensional operators. At this time, we are unaware of any similar results for higher-dimensional operators.

Let us briefly recall the definitions and refer the reader to [48, 62] for textbook treatments. For a compact set $K \subseteq \mathbb{C}$, write $M_1(K)$ for the set of all Borel probability measures supported on K . For $\mu \in M_1(K)$, the (logarithmic) energy of μ is given by

$$\mathcal{E}(\mu) = - \iint \log |x - y| d\mu(x) d\mu(y),$$

with $\mathcal{E}(\mu) = +\infty$ an allowed value. The (logarithmic) capacity of K is

$$\text{Cap}(K) := \sup\{e^{-\mathcal{E}(\mu)} : \mu \in M_1(K)\},$$

where $e^{-\infty} = 0$ by convention. A set with $\text{Cap}(K) = 0$ is known as a *polar set*; such sets play the role of “negligible” sets in potential theory, analogous to the role of measure-zero subsets in measure theory. Polar sets are very small

in the senses that we have discussed herein: they have Hausdorff dimension zero (hence also Lebesgue measure zero) [11]; see also the exercises to [58, Section 3.6]. In terms of these objects, there is a particularly clean lower bound for discrete Schrödinger operators in 1D: one must have $\text{Cap}(\mathcal{S}) \geq 1$. In particular, the spectrum of such a discrete operator may have Hausdorff dimension zero, but it may never be a polar set. This follows from a work of Szegő [65]; see also [56] for a modern discussion. Potential theory for unbounded sets is a good deal more delicate, so it is much harder to phrase things in such a concise fashion. Nevertheless, Eichinger–Lukic have a recent preprint that analyzes Schrödinger operators in $L^2(\mathbb{R})$ from this point of view [21].

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The One World Mathematical Physics Seminar of the IAMP

The executive committee has launched an online seminar of mathematical physics in early May. It joins a series of “One World” seminars initiated by probabilists, and which have propped up because of the coronavirus pandemic.

The first seminar was given on 5 May by Martin Hairer, the 2014 Fields medallist. It has continued every Tuesdays afterwards, with interesting and diverse talks by Solovej, Clavier, Bachmann, Dafermos, Winter. The schedule can be found on the IAMP website at

http://www.iamp.org/page.php?page=page_seminar

The time (UTC 13-14) was chosen so as to accommodate most of our members. The seminar starts at 22:00 in Japan and at 9:00 in the US East Coast. It is less convenient for our members living outside these time zones.

The seminars use the application Zoom and are recorded. They can be watched afterwards on YouTube. The reactions have been very positive. People enjoy listening to the talks and be in contact with other mathematical physicists around the World, including many friends and colleagues. And there are people who are geographically isolated from other mathematical physicists who can attend the talks just as easily than others. Over 250 colleagues have signed up in order to receive the announcements.

A scientific committee has been created which is helping with the selection of speakers and also with deciding the future format. The seminar will pause in August. Whether and how to continue it in September has not been decided yet. Initially the plan was to run it for the duration of the pandemic; but if there is interest, it could continue beyond that. The frequency may be reduced to one seminar every 2-3 weeks, possibly with several speakers. There could also be a longer talk by a senior speaker and a shorter talk by a junior. Comments and suggestions are welcome!

Jan Dereziński and Daniel Ueltschi

Call for nominations for the 2021 IAMP Early Career Award

The IAMP Executive Committee calls for nominations for the 2021 IAMP Early Career Award. The prize was instituted in 2008 and will be awarded for the fourth time at the ICMP in Geneva, Switzerland in July, 2021.

The Early Career Award is given in recognition of a single achievement in mathematical physics. The total prize value amounts to 3000 Euro and is reserved for scientists whose age in years since birth on July 31 of the year of the Congress is less than 35.

The nomination should include the name of the candidate accompanied by a brief characterization of the work singled out for the prize. Members of the IAMP should send their nomination or nominations to the President (president@iamp.org) and to the Secretary (secretary@iamp.org). A list of previous winners and the details of the award selection process can be found at <http://www.iamp.org>.

Nominations should be made not later than on January 31, 2021.

Call for nominations for the 2021 IUPAP Young Scientist Prize in Mathematical Physics

IUPAP Commission C18 (Mathematical Physics) calls for nominations for the IUPAP Young Scientist Prize in Mathematical Physics.

The prize recognizes exceptional achievements in mathematical physics by scientists at relatively early stages of their careers. It is awarded triennially to at most three young scientists satisfying the following criteria:

- The recipients of the awards in a given year should have a maximum of 8 years of research experience (excluding career interruptions) following their PhD on January 1 of that year (in this case that is 2021).
- The recipients should have performed original work of outstanding scientific quality in mathematical physics.
- Preference may be given to young mathematical physicists from developing countries. The awards will be presented at the ICMP in August, 2021 in Geneva.

A nomination should include a brief description of the achievements of the candidate that support the nomination, a CV, and a list of publications (or current links to that information online).

Please submit nominations to

Rajesh Gopakumar (secretary, rajesh.gopakumar@icts.res.in)

Alain Joye (vice-chair, alain.joye@univ-grenoble-alpes.fr)

Bruno Nachtergaele (chair, bxn@math.ucdavis.edu).

The deadline for nominations is September 30, 2020.

For further information, including past recipients, see

<https://iupap.org/commissions/c18-mathematical-physics/c18-awards/>.

Call for Nominations for the 2021 Henri Poincaré Prize

Dear colleagues,

I would like to draw your attention to our call for nominations for the 2021 Henri Poincaré Prize; the deadline for nominations is September 30, 2020.

Please also spread the word among your colleagues who are not IAMP members.

The nomination should include the following:

- Description of the scientific work of the nominee emphasising her/his key contributions
- A recent CV of the nominee
- A proposed citation should the nominee be selected for an award

Nominations should be sent to the President (president@iamp.org) or to the Secretary (secretary@iamp.org) of the IAMP.

Note that members of the Executive Committee of the Association

http://www.iamp.org/page.php?page=page_about

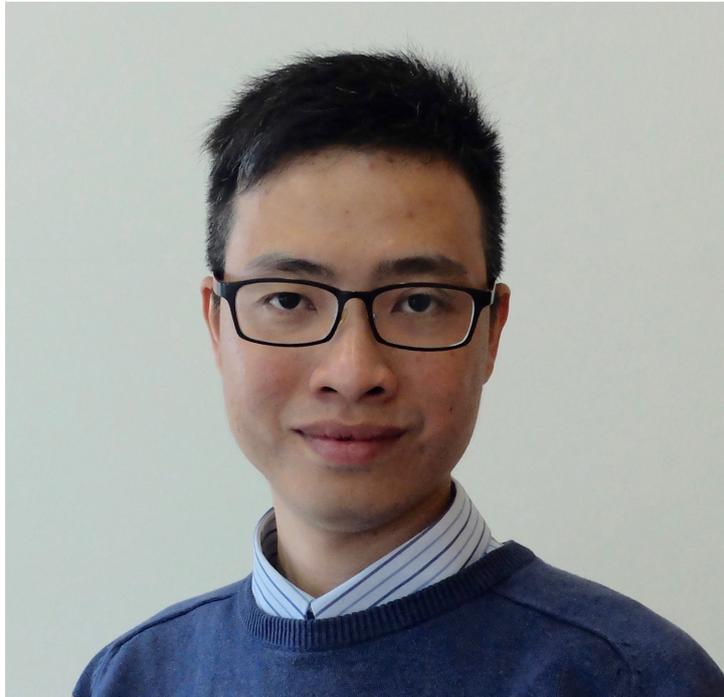
are not eligible for nomination for the prize. For past winners see

http://www.iamp.org/page.php?page=page_prize_poincare.

Best regards

Robert Seiringer
(IAMP president)

Phan Thành Nam wins EMS prize



The *News Bulletin* has learned that one of our members, Phan Thành Nam, has recently received an EMS Prize. Ten of these prizes are awarded once every four years by the European Mathematical Society “to young researchers not older than 35 years, of European nationality or working in Europe, in recognition of excellent contributions in mathematics.”

On behalf of our community,

Congratulations, Nam!

The EMS news release is at

<https://euro-math-soc.eu/news/20/05/8/prize-winners-announced>

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. PROF. SUNIL MAHARAJ, University of KwaZulu-Natal, Durban, South Africa.
2. DR. MARCO BENINI, University of Genova, Italy.
3. DR. NIKOLAI LEOPOLD, University of Basel, Switzerland.
4. PROF. JOHN WALKUP, California State University, Fresno, USA.

Recent conference announcements

[Ari Laptev's 70th birthday conference](https://www.iadm.uni-stuttgart.de/en/ari-laptevs-70-birthday-conference/)

Virtual conference.

<https://www.iadm.uni-stuttgart.de/en/ari-laptevs-70-birthday-conference/>

Open positions

Postdoctoral position at EPFL

The Chair of Analysis and Mathematical Physics at the Institute of Mathematics at EPFL (Ecole polytechnique fédérale de Lausanne) welcomes applications for a postdoctoral position.

The appointment will be primarily based on excellent research potential in mathematical physics performed by analytical methods. Areas of research interest at the chair, which will be headed by Prof. Marius Lemm as of September 1, 2020, are the spectral theory of Schrödinger operators, homogenization theory, quantum spin systems, quantum information theory, and random matrices.

The teaching load for the postdoctoral fellow consists of serving as teaching assistant for 1 course per semester and advising a small number of undergraduate theses.

English is the main language spoken at EPFL, and Lausanne is highly international, so no French language skills are required. EPFL is committed to being an equal opportunity employer.

The start date is flexible and is expected to lie between October 1, 2020 and June 1, 2021. The contract is initially for 1 year and renewable up to 4 years. Review of applications will begin on September 1, 2020. Applications received after this date will still be considered if the position is not filled yet.

For additional information, including how to apply for this position, visit either of the following websites:

<https://recruiting.epfl.ch/Vacancies/1363/Description/2>

<https://www.mathjobs.org/jobs/list/16012>

Doctoral researchers in Karlsruhe (Germany)

The Collaborative Research Center “Wave phenomena – analysis and numerics” (CRC 1173), is currently seeking to recruit, as soon as possible, limited to three years, four Doctoral Researchers (f/m/d) (75-100%).

The CRC has been funded by the German Research Foundation (DFG) since 2015. Its goal is to analytically understand, numerically simulate, and eventually manipulate wave propagation under realistic scenarios by intertwining analysis and numerics.

We seek ambitious doctoral researchers to work in the area of wave phenomena, in particular on the analysis or numerics of partial differential equations. For detailed information on the projects please visit <http://www.waves.kit.edu/joboffers.php>

The following qualifications are required: Excellent Master or an equivalent degree in Mathematics, strong background in analysis or numerical analysis of partial differential equations, we expect good writing and oral communication skills in English along with the ability to work independently within an international team, for a project with emphasis on numerics, programming skills in MATLAB, Python, or C/C++ are highly beneficial.

Applications should include a cover letter, a curriculum vitae, a statement of research interest, contact information for two referees, and copies of degree certificate(s).

We prefer to balance the number of employees (f/m/d). Therefore, we kindly ask female applicants to apply for this job. If qualified, severely disabled persons will be preferred.

Please apply online (<https://www.waves.kit.edu/joboffers.php>) until 20. August 2020 using the vacancy number 3647/2020 and reference number 8. Personnel support is provided by Ms. Brückner, Personalservice, Karlsruhe Institute of Technology (KIT), Campus Süd, Kaiserstraße 12, 76131 Karlsruhe. For further information, please contact Ms. Laurette Lauffer, phone +49(0)721/608-42061 or Laurette.lauffer@kit.edu.

Postdoctoral researcher in Karlsruhe (Germany)

The Collaborative Research Center “Wave phenomena – analysis and numerics” (CRC 1173), is currently seeking to recruit, as soon as possible, limited to two years, a Postdoctoral Researcher (f/m/d), associated with the project “Non-linear Helmholtz equations and systems”.

The CRC has been funded by the German Research Foundation (DFG) since 2015. Its goal is to analytically understand, numerically simulate, and eventually manipulate wave propagation under realistic scenarios by intertwining analysis and numerics.

The Associated Project “Nonlinear Helmholtz Equations and Systems” aims to establish a solution theory describing oscillatory solutions of Nonlinear Helmholtz Equations and time harmonic Maxwell’s Equations in inhomogeneous media. One of the central tasks is the derivation of new Limiting Absorption Principles and resolvent estimates for the linear operators under investigation as well as their applications to nonlinear problems.

We seek an ambitious postdoctoral researcher with strong research interest in these or related research fields, which is preferably demonstrated by outstanding publications or preprints. Engagement in teaching is expected.

The following qualifications are required: Excellent PhD thesis and expertise in partial differential equations, we expect good writing and oral communication skills in English along with the ability to work independently within an international team.

Applications should include a cover letter, a curriculum vitae, links to your publications and your PhD thesis, a statement of research interest (not more than 2 pages), contact information for two referees, and copies of degree certificate(s) in one pdf file.

We prefer to balance the number of employees (f/m/d). Therefore, we kindly ask female applicants to apply for this job. If qualified, severely disabled persons will be preferred.

Please apply online (<https://www.waves.kit.edu/joboffers.php>) until August 31, 2020, using the vacancy number 3660/2020 and reference number 8. Personnel support is provided by Ms. Brückner, Personalservice, Karlsruhe Institute of Technology (KIT), Campus Süd, Kaiserstraße 12, 76131 Karlsruhe.

For further information, please contact

Dr. Rainer Mandel, phone +49(0)721-608-46178 or

Ms. Laurette Lauffer, phone +49(0)721/608-42061, Laurette.lauffer@kit.edu.

Several PhD positions in Bielefeld, Germany

The German–Korean International Research Training Group (IRTG) 2235 “Searching for the regular in the irregular: Analysis of singular and random systems” funded by the Deutsche Forschungsgemeinschaft (DFG), offers several PhD positions at Bielefeld University, starting January 1, 2021.

The IRTG is a joint research program established by the Faculty of Mathematics at Bielefeld University, Germany, and the Department of Mathematical Sciences at Seoul National University, South Korea. In a truly international and competitive environment, doctoral students study singular and random systems. The IRTG concentrates on advanced techniques from the mathematical field of analysis including geometric, stochastic and numerical analysis together with latest developments in neighboring fields such as mathematical physics and probability theory. The focus is on the mathematical analysis of problems which generically exhibit singular features or randomness. The topics include stochastic dynamics, stochastic partial differential equations, machine learning, nonlinear wave equations, integrodifferential equations, random matrices, generalized Dirichlet forms, analysis on manifolds and fractal metric spaces. The qualification program of the IRTG includes a structured course program in English and a six-months research stay at Seoul National University. PhD students

of the IRTG will be offered an E 13 TV-L (75%) position for up to 3 years.

Candidates should hold an excellent Master degree (or an equivalent degree) in Mathematics, Mathematical Physics or a neighboring field. Moreover, communication skills are important for a successful integration into the IRTG and the participating departments. Proficiency in English is required whereas prior knowledge of German is not mandatory.

Application process: Details about the application process can be found on the website of the IRTG:

https://irtg.math.uni-bielefeld.de/doctoral_positions/open2020

Application deadline: August 23, 2020

Contact: Dr. Claudia Köhler, Academic Coordinator of the IRTG 2235,
irtg@math.uni-bielefeld.de.

Professor or associate/assistant professor in mathematical physics at the University of Helsinki

The Department of Mathematics and Statistics at the Faculty of Science of the University of Helsinki invites applications for a Professor or Associate/Assistant professor in Mathematical physics.

A successful applicant must have a strong track record in mathematically rigorous aspects of mathematical physics. Depending on her or his qualifications and career stage, the appointment can be made to a permanent full professorship or a fixed-term assistant/associate professorship (tenure track).

Detailed announcement and instructions can be found at

www.helsinki.fi/en/open-positions/professor-or-assistantassociate-professor-in-mathematical-physics

The deadline for applications is August 15, 2020.

For more information on these positions and for an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

Benjamin Schlein (IAMP Secretary)

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