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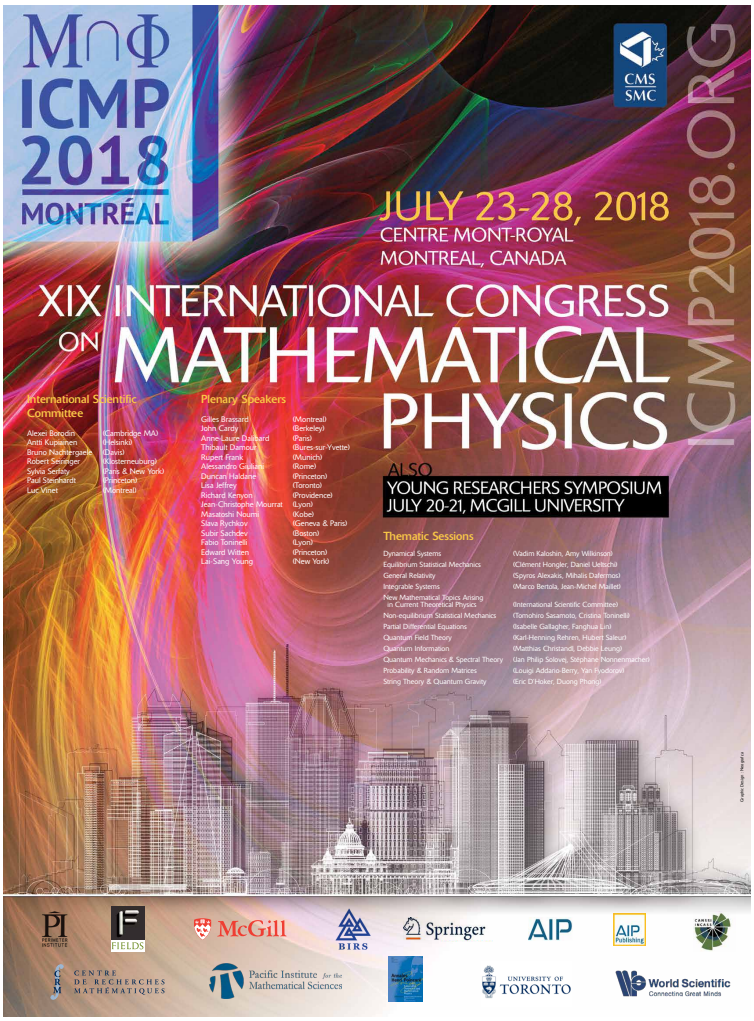

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Contents

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Nodal domains, spectral minimal partitions, and their relation to Aharonov-Bohm operators

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This survey is a short version of a chapter written by the first two authors in the book [66] (where more details and references are given) but we have decided here to put more emphasis on the role of the Aharonov-Bohm operators which appear to be a useful tool coming from physics for understanding a problem motivated either by spectral geometry or dynamics of population. Similar questions appear also in Bose-Einstein theory. Finally some open problems which might be of interest are mentioned.

1 Introduction

In this survey, we mainly consider the Dirichlet realization of the Laplacian operator in Ω, when Ω is a bounded domain in $\mathbb{R}^2$ with piecewise-$C^{1,+}$ boundary (domains with corners or cracks[1] also permitted). This operator will be denoted by $H(\Omega)$. We would like to analyze the connections between the nodal domains of the eigenfunctions of $H(\Omega)$ and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground-state energy of the Dirichlet realization of the Laplacian $H(D_i)$ is minimal. This problem can be seen as a strong competition limit of segregating species in population dynamics (see [36], [42] and references therein). Similar questions appear also in the analysis of the segregation and the symmetry breaking of a two-component condensate in the Bose-Einstein theory (see [80] and references therein), with $\Omega = \mathbb{R}^2$ and $H(\Omega)$ replaced by the harmonic oscillator or more generally by a Schrödinger operator.

To be more precise, we start from the following weak notion of partition:
A **partition** *(or k-partition for indicating the cardinality of the partition) of Ω is a family $\mathcal{D} = \{D_i\}_{1\le i\le k}$ of k mutually disjoint sets in Ω (with $k \ge 1$ an integer).*
If we denote by $\mathfrak{O}_k(\Omega)$ the set of partitions of Ω where the D_i's are domains (*i.e.* open and connected), we introduce the energy of the partition:

$$\Lambda(\mathcal{D}) = \max_{1\le i\le k} \lambda(D_i), \tag{1}$$

where $\lambda(D_i)$ is the ground state energy (*i.e.* the lowest eigenvalue) of $H(D_i)$. The optimal problem we are interested in is the determination, for any integer $k \ge 1$, of

$$\mathfrak{L}_k = \mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k(\Omega)} \Lambda(\mathcal{D}). \tag{2}$$

We can also consider the case of a two-dimensional Riemannian manifold and the Laplacian is then the Laplace-Beltrami operator. We say that (φ, λ) is a spectral pair for $H(\Omega)$ if λ is an eigenvalue of the Dirichlet Laplacian $H(\Omega)$ on Ω and $\varphi \in E(\lambda) \setminus \{0\}$, where $E(\lambda)$ denotes the eigenspace attached to λ. We denote by $\{\lambda_n(\Omega), n \ge 1\}$ the nondecreasing sequence of eigenvalues of $H(\Omega)$ and by $\{\varphi_n, n \ge 1\}$ some associated orthonormal basis of eigenfunctions. The

[1] For example a square with a segment removed. By $C^{1,+}$, we mean $C^{1,\alpha}$ for some $\alpha > 0$.

ground-state φ_1 can be chosen to be strictly positive in Ω, but the other excited eigenfunctions φ_n must have zero sets. Here we recall that for $\varphi \in C^0(\overline{\Omega})$, the nodal set (or zero set) of φ is defined by :

$$\mathcal{N}(\varphi) = \overline{\{\mathbf{x} \in \Omega \mid \varphi(\mathbf{x}) = 0\}}\,. \tag{3}$$

In the case when φ is an eigenfunction of the Laplacian, the $\mu(\varphi)$ components of $\Omega \setminus \mathcal{N}(\varphi)$ are called the nodal domains of φ and define naturally a partition of Ω by $\mu(\varphi)$ open sets, which is called a **nodal partition**.

Our main goal is to discuss the links between the partitions of Ω associated with these eigenfunctions and the minimal partitions of Ω. We will also describe how these minimal partitions can also be seen as a nodal partition of an eigenfunction of a suitable Aharonov-Bohm operator. There is of course a strong relation between the analysis of nodal sets of eigenfunctions and nodal domains. Here the surveys of S. Zelditch, [86, 88] show the growing importance of this field. There mostly the case of the Laplace Beltrami operator on smooth manifolds is reviewed. The results about nodal sets of general Schrödinger operators are rather scattered and we are not able to find a good survey for this. We want to stress that these topics represent only a small part of this and related areas in physics and mathematics. We just mention some of the surveys and papers which demonstrate that this field is huge and there are many different communities which in many cases, unfortunately, interact only little. Two instructive reviews are [49, 69] written by mathematical physicists where questions about quantum chaos, nodal domain statistics and many other topics are discussed. Many questions about nodal domains and partitions have their counterparts in quantum graphs and spectral graph theory, see for instance [44, 16, 20].

2 Nodal partitions

2.1 On the local structure of nodal sets

We refer for this section to the survey of P. Bérard [10]. We recall that, if φ is an eigenfunction associated with λ and D is one of its nodal domains, then the restriction of φ to D belongs to $H_0^1(D)$ and is an eigenfunction of the Dirichlet Laplacian in D. Moreover, λ is the ground state energy in D.

Using [18, 40], we can prove that nodal sets are regular in the sense that:

- The singular points $\mathbf{x}_0$ on the nodal lines are isolated.
- At the singular points, an even number of half-lines meet with equal angle.
- At the boundary, this is the same as adding the tangent line in the picture.

The notion of regularity will be defined later for general partitions.

2.2 Courant's theorem and Courant-sharp eigenvalues

The following theorem was established by R. Courant [43] in 1923 for the Laplacian with Dirichlet or Neumann conditions.

Theorem 2.1 (Courant)**.** *The number of nodal components of the k-th eigenfunction is not greater than k.*

We say that a spectral pair (φ, λ) is Courant-sharp if $\lambda = \lambda_k$ and u has k nodal domains. We say that an eigenvalue λ_k is Courant-sharp if there exists an eigenfunction φ associated with λ_k such that (φ, λ_k) is a Courant-sharp spectral pair.

Whereas the Sturm-Liouville theory shows that in dimension 1 all the spectral pairs are Courant-sharp, we will see below that in higher dimension, the Courant-sharp situation can only occur for a finite number of eigenvalues.

2.3 Pleijel's theorem

Pleijel proved the following theorem in 1956 [78]:

Theorem 2.2 (Weak Pleijel's theorem)**.** *If the dimension is ≥ 2, there are only finitely many Courant-sharp eigenvalues of the Dirichlet Laplacian.*

This theorem is the consequence of a more precise theorem which gives a link between Pleijel's theorem and partitions. For describing this result and its proof, we first recall the Faber-Krahn inequality:

Theorem 2.3 (Faber-Krahn inequality)**.** *For any domain $D \subset \mathbb{R}^2$, we have*

$$|D|\, \lambda(D) \geq \lambda(\bigcirc)\,, \tag{4}$$

where $|D|$ denotes the area of D and $\bigcirc$ is the disk of unit area $\mathcal{B}\Big(0, \frac{1}{\sqrt{\pi}}\Big)$.

Note that improvements can be useful when D is "far" from a disk. It is then interesting to have a lower bound for $|D|\, \lambda(D) - \lambda(\bigcirc)$. We refer for example to [33] and [53]. These ideas are behind recent improvements by Steinerberger [83], Bourgain [32] and Donnelly [45] of the strong Pleijel theorem below. Its proof is indeed enlightning. First, by summation of Faber-Krahn inequalities (4) applied to each D_i and having in mind the definition of the energy, we deduce that for any open partition $\mathcal{D}$ of Ω we have

$$|\Omega|\, \Lambda(\mathcal{D}) \geq \sharp(\mathcal{D})\, \lambda(\bigcirc)\,, \tag{5}$$

where $\sharp(\mathcal{D})$ denotes the number of elements of the partition.
Secondly, we implement Weyl's formula for the counting function of the Laplacian which reads in dimension d

$$N(\lambda) \sim \frac{\omega_d}{(2\pi)^d} |\Omega| \lambda^{\frac{d}{2}}, \qquad \text{as } \lambda \to +\infty\,, \tag{6}$$

where ω_d denotes the volume of a ball of radius 1 in $\mathbb{R}^d$ and $|\Omega|$ the volume of Ω. This leads to:

Theorem 2.4 (Strong Pleijel theorem)**.** *Let φ_n be an eigenfunction of $H(\Omega)$ associated with $\lambda_n(\Omega)$. Then*

$$\limsup_{n \to +\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{4\pi}{\lambda(\bigcirc)}\,. \tag{7}$$

Remark 2.5. *It is natural (see an important motivation for minimal partitions in the next section) to determine the Courant-sharp situation for some examples. This kind of analysis was initiated by Å. Pleijel for the square and continued in [62] for the disk and some rectangles (rational or irrational). We recall that, according to Theorem 2.2, there are a finite number of Courant-sharp eigenvalues. The point is to quantify this number or to find lower bounds or upper bounds for the largest integer n such that $\lambda_{n-1} < \lambda_n$ with λ_n Courant-sharp. This involves an explicit control of the remainder in Weyl's formula and then a case by case analysis of the remaining eigenspaces.*

Other domains have been analyzed by various subgroups of the set of authors (Band, Bérard, Bersudsky, Charron, Fajman, Helffer, Hoffmann-Ostenhof, Kiwan, Léna, Leydold, Persson-Sundqvist, Terracini, ..., see [66] for the references): the square and the annulus for the Neumann Laplacian, the sphere, the irrational and equilateral torus, the triangle (equilateral, hemi-equilateral, right angled isosceles), Neumann 2-rep-tiles, the cube, the ball, the 3D-torus.

Remark 2.6. *Pleijel's Theorem extends to bounded domains in $\mathbb{R}^d$, and more generally to compact d-manifolds with boundary, with a constant $\gamma(d) < 1$ replacing $4\pi/\lambda(\bigcirc)$ in the right-hand side of (7) (see Peetre [77], Bérard-Meyer [14]). This constant is independent of the geometry. It is also true for the Neumann Laplacian in a piecewise analytic bounded domain in $\mathbb{R}^2$ (see [79] whose proof is based on a control of the asymptotics of the number of boundary points belonging to the nodal sets of the eigenfunction associated with λ_k as $k \to +\infty$, a difficult result proved by Toth-Zelditch [84]). C. Léna [74] gets the same result for C^2 domains, without any condition on the dimension, through a very nice decomposition of the nodal domains. In [13, 85], the authors determine an upper bound for Courant-sharp eigenvalues, expressed in terms of simple geometric invariants of Ω. Finally, it is expected that Pleijel's theorem can be extended to Schrödinger operators $-\Delta + V$, either for the negative spectrum if $V \to 0$ as $|\mathbf{x}| \to +\infty$ (for instance the potential of the Hydrogen atom) or for the whole spectrum if $V \to +\infty$. This has been proved for the harmonic oscillator by P. Charron and for radial potentials by Charron-Helffer-Hoffmann-Ostenhof (see [39] and references therein).*

3 Minimal spectral partitions

3.1 Definitions

Spectral minimal partitions were introduced in [62] within a more general class depending on $p \in [1, +\infty]$ ($p = 1$ and $p = +\infty$ being physically the most interesting). For any integer $k \geq 1$ and $p \in [1, +\infty[$, we define the p-energy of a k-partition $\mathcal{D} = \{D_i\}_{1 \leq i \leq k}$ by

$$\Lambda_p(\mathcal{D}) = \Big(\frac{1}{k}\sum_{i=1}^{k} \lambda(D_i)^p\Big)^{\frac{1}{p}} . \tag{8}$$

The associated optimization problem is written

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D} \in \mathfrak{D}_k(\Omega)} \Lambda_p(\mathcal{D}) , \tag{9}$$

and we call a k-partition with p-energy $\mathfrak{L}_{k,p}(\Omega)$ a *p-minimal k-partition*. For $p = +\infty$, we write $\Lambda_\infty(\mathcal{D}) = \Lambda(\mathcal{D})$ and $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$.

The analysis of the properties of minimal partitions leads us to introduce two notions of regularity:

– A partition $\mathcal{D} = \{D_i\}_{1\le i\le k}$ of Ω in $\mathfrak{O}_k(\Omega)$ is called **strong** if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,.$$

– It is called **nice** if

$$D_i = \mathrm{Int}\,(\overline{D_i}) \cap \Omega\,,$$

for any $1 \le i \le k$.

In Figure 4, only the fourth picture gives a nice partition. Attached to a strong partition, we associate the boundary set

$$\partial\mathcal{D} = \overline{\cup_i\,(\Omega \cap \partial D_i)}\,, \tag{10}$$

which plays the role of the nodal set (in the case of a nodal partition).
To go further, we introduce the set $\mathfrak{O}_k^{\mathsf{reg}}(\Omega)$ of **regular** partitions, which should satisfy the following properties:

(i) Except at finitely many distinct $\mathbf{x}_i \in \Omega\cap\partial\mathcal{D}$ in the neigborhood of which $\partial\mathcal{D}$ is the union of $\nu(\mathbf{x}_i)$ smooth curves ($\nu(\mathbf{x}_i) \ge 2$) with one end at $\mathbf{x}_i$, $\partial\mathcal{D}$ is locally diffeomorphic to a regular curve.

(ii) $\partial\Omega \cap \partial\mathcal{D}$ consists of a (possibly empty) finite set of points $\mathbf{y}_j$. Moreover $\partial\mathcal{D}$ is near $\mathbf{y}_j$ the union of $\rho(\mathbf{y}_j)$ distinct smooth half-curves which hit $\mathbf{y}_j$.

(iii) $\partial\mathcal{D}$ has the equal angle meeting property, that is the half curves cross with equal angle at each singular interior point of $\partial\mathcal{D}$ and also at the boundary together with the tangent to the boundary.

We denote by $X(\partial\mathcal{D})$ the set corresponding to the points $\mathbf{x}_i$ introduced in (i) and by $Y(\partial\mathcal{D})$ corresponding to the points $\mathbf{y}_j$ introduced in (ii).

This notion of regularity for partitions is very close to what we have observed for the nodal partition of an eigenfunction. The main difference is that, in the nodal case, there is always an even number of half-lines meeting at an interior singular point.

3.2 Bipartite partitions

Two sets D_i, D_j of the partition $\mathcal{D}$ are neighbors and we write $D_i \sim D_j$, if $D_{ij} = \mathrm{Int}\,(\overline{D_i \cup D_j})\setminus \partial\Omega$ is connected. A regular partition is bipartite if it can be colored by two colors (two neighbors always having two different colors).
Nodal partitions are the main examples of bipartite partitions. Note that in the case of a simply connected planar domain, we know by graph theory that, if for a regular partition all the $\nu(\mathbf{x}_i)$ are even, then the partition is bipartite. This is no more the case on an annulus or on a surface.

3.3 Main properties of minimal partitions

The following theorem has been proved by Conti-Terracini-Verzini (existence) and Helffer–Hoffmann-Ostenhof–Terracini (regularity) ([62] and references therein):

Theorem 3.1. *For any k, there exists a minimal k-partition which is strong and regular. Moreover any minimal k-partition has a strong and regular representative*[2]*. The same result holds for the p-minimal k-partition problem with $p \in [1, +\infty)$.*

Note that the regularity property implies that a minimal partition is nice.

When $p = +\infty$, minimal spectral partitions have two important properties. Let $\mathcal{D} = \{D_i\}_{1\leq i\leq k}$ be a minimal k-partition. Then

- The minimal partition $\mathcal{D}$ is a spectral equipartition, *i.e.* $\mathfrak{L}_k(\Omega) = \lambda(D_i), \forall 1 \leq i \leq k$.
- For any pair of neighbors $D_i \sim D_j$, $\lambda_2(D_{ij}) = \mathfrak{L}_k(\Omega)$.

For the first property, this can be understood, once regularity is obtained, by pushing the boundary and using the Hadamard formula [65]. For the second property, we can observe that $\{D_i, D_j\}$ is necessarily a minimal 2-partition of D_{ij}. This leads to what we call the *pair compatibility condition*.

Note that in the proof of Theorem 3.1, one obtains on the way a useful construction. Attached to each D_i, there is a distinguished ground state u_i such that $u_i > 0$ in D_i and such that for each pair of neighbors $\{D_i, D_j\}$, $u_i - u_j$ is the second eigenfunction of the Dirichlet Laplacian in D_{ij}.

Let us now establish two important properties concerning the monotonicity (according to k or the domain Ω):

- For any $k \geq 1$, we have $\mathfrak{L}_k(\Omega) < \mathfrak{L}_{k+1}(\Omega)$.
- If $\Omega \subset \widetilde{\Omega}$, then $\mathfrak{L}_k(\widetilde{\Omega}) \leq \mathfrak{L}_k(\Omega)$ for any $k \geq 1$.

3.4 Minimal spectral partitions and Courant-sharp property

A natural question is whether a minimal partition of Ω is a nodal partition. We have first the following converse theorem (see [62]):

Theorem 3.2. *If a minimal partition is bipartite, it is a nodal partition.*

Proof. Combining the bipartite assumption and the consequence of the pair compatibility condition mentioned after Theorem 3.1, it is immediate to construct some $\varphi \in H_0^1(\Omega)$ such that

$$\varphi_{|D_i} = \pm\varphi_i, \quad \forall 1 \leq i \leq k, \qquad \text{and} \qquad -\Delta\varphi = \mathfrak{L}_k(\Omega)\varphi \quad \text{in } \Omega \setminus X(\partial\mathcal{D}).$$

A capacity argument shows that $-\Delta\varphi = \mathfrak{L}_k(\Omega)\varphi$ in all Ω and hence φ is an eigenfunction of $H(\Omega)$ whose nodal set is $\partial\mathcal{D}$. □

[2]possibly after a modification of the open sets of the partition by capacity 0 subsets.

The next question is then to determine how general the previous situation is. Surprisingly, this only occurs in the so called Courant-sharp situation. For the statement, we need another spectral sequence. For any $k \geq 1$, we denote by $L_k(\Omega)$ (or L_k if there is no confusion) the smallest eigenvalue (if any) for which there exists an eigenfunction with k nodal domains. We set $L_k(\Omega) = +\infty$ if there is no eigenfunction with k nodal domains. In general, one can show, as an easy consequence of the max-min characterization of the eigenvalues, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega)\,. \tag{11}$$

The following important theorem (due to [62]) gives the full picture of the equality cases:

Theorem 3.3. *Suppose $\Omega \subset \mathbb{R}^2$ is regular. If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega)$, then*

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega)\,. \tag{12}$$

In addition, there exists a Courant-sharp eigenfunction associated with $\lambda_k(\Omega)$.

Sketch of the proof. It is easy to see, using a variation of the proof of Courant's theorem, that the equality $\lambda_k = \mathfrak{L}_k$ implies (12). Hence the difficult part is to get (12) from the assumption that $L_k = \mathfrak{L}_k = \lambda_{m(k)}$, that is to prove that $m(k) = k$. This involves a construction of an exhaustive family $\{\Omega(t),\ t \in (0,1)\}$, interpolating between $\Omega(0) := \Omega \setminus \mathcal{N}(\psi_k)$ and $\Omega(1) := \Omega$, where ψ_k is an eigenfunction corresponding to L_k such that its nodal partition is a minimal k-partition. This family is obtained by cutting small intervals in each regular component of $\mathcal{N}(\psi_k)$. L_k is an eigenvalue common to all $H(\Omega(t))$, but its labeling changes between $t = 0$ and $t = 1$ at some t_0 where the multiplicity of L_k should increase. By a tricky argument which is not detailed here, we get a contradiction. □

Similar results hold in the case of compact Riemannian surfaces when considering the Laplace-Beltrami operator. Typical cases are analyzed like $\mathbb{S}^2$ in [64] and $\mathbb{T}^2$ in [71]. In the case of dimension 3, let us mention that Theorem 3.3 is proved in [63]. The complete analysis of the topology of minimal partitions in higher dimension is not achieved as for the two-dimensional case.

3.5 Topology of regular partitions

Euler's formula for regular partitions. In the case of planar domains, one can use a variant of Euler's formula in the following form (see [68, 11]).

Proposition 3.4. *Let Ω be an open set in $\mathbb{R}^2$ with piecewise $C^{1,+}$ boundary and $\mathcal{D}$ be a k-partition with $\partial\mathcal{D}$ the boundary set. Let b_0 be the number of components of $\partial\Omega$ and b_1 be the number of components of $\partial\mathcal{D} \cup \partial\Omega$. Denote by $\nu(\mathbf{x}_i)$ and $\rho(\mathbf{y}_i)$ the numbers of curves ending at $\mathbf{x}_i \in X(\partial\mathcal{D})$, respectively $\mathbf{y}_i \in Y(\partial\mathcal{D})$. Then*

$$k = 1 + b_1 - b_0 + \sum_{\mathbf{x}_i \in X(\partial\mathcal{D})} \Big(\frac{\nu(\mathbf{x}_i)}{2} - 1\Big) + \frac{1}{2} \sum_{\mathbf{y}_i \in Y(\partial\mathcal{D})} \rho(\mathbf{y}_i)\,. \tag{13}$$

This can be applied together with other arguments to determine upper bounds for the number of singular points of minimal partitions. There is a corresponding result for compact manifolds involving the Euler characteristics.

Application to regular 3-partitions. As an application of the Euler formula, we can describe (see [57]) the possible "topological" types of non-bipartite minimal 3-partitions when Ω is a simply-connected domain in $\mathbb{R}^2$. Then the boundary set $\partial\mathcal{D}$ has one of the following properties:

[a] one interior singular point $\mathbf{x}_0 \in \Omega$ with $\nu(\mathbf{x}_0) = 3$, three points $\{\mathbf{y}_i\}_{1\leq i\leq 3}$ on the boundary $\partial\Omega$ with $\rho(\mathbf{y}_i) = 1$;

[b] two interior singular points $\mathbf{x}_0,\ \mathbf{x}_1 \in \Omega$ with $\nu(\mathbf{x}_0) = \nu(\mathbf{x}_1) = 3$ and two boundary singular points $\mathbf{y}_1,\ \mathbf{y}_2 \in \partial\Omega$ with $\rho(\mathbf{y}_1) = 1 = \rho(\mathbf{y}_2)$;

[c] two interior singular points $\mathbf{x}_0,\ \mathbf{x}_1 \in \Omega$ with $\nu(\mathbf{x}_0) = \nu(\mathbf{x}_1) = 3$ and no singular point on the boundary.

This helps us to analyze (with some success) the minimal 3-partitions with some topological type. We actually do not know any example where the minimal 3-partitions are of type [b] and [c] (see numerical computations in [27] for the square and the disk, [28] for circular sectors and see [25] for complements in the case of the disk).

Upper bound for the number of singular points. Euler's formula also implies

Proposition 3.5. *Let $\mathcal{D}$ be a minimal k-partition of a simply connected domain Ω with $k \geq 2$. Let $X^{\mathsf{odd}}(\partial\mathcal{D})$ be the subset among the interior singular points $X(\partial\mathcal{D})$ for which $\nu(\mathbf{x}_i)$ is odd. Then the cardinality of $X^{\mathsf{odd}}(\partial\mathcal{D})$ satisfies*

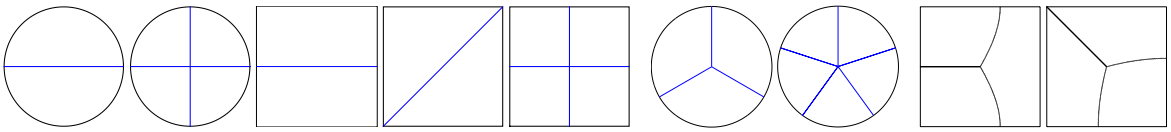
$$\sharp X^{\mathsf{odd}}(\partial\mathcal{D}) \leq 2k - 4\,. \tag{14}$$

In the case of $\mathbb{S}^2$, one can prove that a minimal 3-partition is not nodal (the second eigenvalue has multiplicity 3), and as a step towards a characterization, one can show that non-nodal minimal partitions have necessarily two singular *triple points* (*i.e.* with $\nu(\mathbf{x}) = 3$). If we assume, for some $k \geq 12$, that a minimal k-partition has only singular triple points and consists only of (spherical) pentagons and hexagons, then Euler's formula in its historical version for convex polyedra $V - E + F = \chi(\mathbb{S}^2) = 2$ (where F is the number of faces, E the number of edges and V the number of vertices) implies that the number of pentagons is 12. It has been proved by Soave-Terracini [82, Theorem 1.12] that

$$\mathfrak{L}_3(\mathbb{S}^d) = \frac{3}{2}\left(d + \frac{1}{2}\right).$$

3.6 Examples of minimal k-partitions

When Ω is a disk or a square, one can show the minimal k-partition are nodal only for $k = 1, 2, 4$ (see Figure 1a for $k = 2, 4$). For other k's, the question is open. Numerical simulations in [24, 25] permit to exhibit candidates to be minimal k-partition of the disk for $k = 3, 5$.



(a) Minimal k-partitions, $k = 2, 4$. (b) 3- and 5-partition. (c) $\mathcal{D}^{\mathsf{perp}}$ and $\mathcal{D}^{\mathsf{diag}}$.

Figure 1: Partitions of the disk and the square.

Nevertheless we have no proof that the minimal 3-partition of the disk is the "Mercedes star" (see Figure 1b).

Let us now discuss the 3-partitions of a square. It is not difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. Numerical computations[3] in [27] produce natural candidates for a symmetric minimal 3-partition. Two candidates $\mathcal{D}^{\mathsf{perp}}$ and $\mathcal{D}^{\mathsf{diag}}$ are obtained numerically by choosing the symmetry axis (perpendicular bisector or diagonal line) and represented in Figure 1c. Numerics suggests that there is no candidate of type [b] or [c], that the two candidates $\mathcal{D}^{\mathsf{perp}}$ and $\mathcal{D}^{\mathsf{diag}}$ have the same energy, and that the center is the unique singular point of the partition inside the square. Once this last property is accepted, one can perform the spectral analysis of an Aharonov-Bohm operator (see Section 4) with a pole at the center. This point of view is explored numerically in a rather systematic way by Bonnaillie-Noël–Helffer [24] and theoretically by Noris-Terracini [76] (see also [30]). This could explain why the two partitions $\mathcal{D}^{\mathsf{perp}}$ and $\mathcal{D}^{\mathsf{diag}}$ have the same energy. Moreoever this suggests that there is a continuous fam-

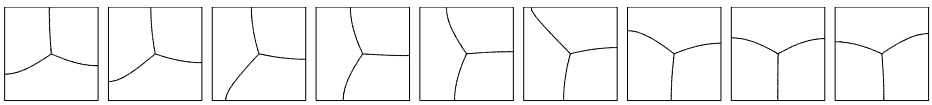


Figure 2: A continuous family of 3-partitions with the same energy.

ily of minimal 3-partitions of the square. This is illustrated in Figure 2. In the formalism of the Aharonov-Bohm operator, the basic remark is that this operator has an eigenvalue of multiplicity 2 when the pole is at the center. We refer to [26, 24] for further discussion.

4 Aharonov-Bohm operators and minimal partitions

The introduction of Aharonov-Bohm operators in this context is an example of "physical mathematics". There is no magnetic field in our problem and it is introduced artificially. But the idea comes from [56], which was motivated by a problem in superconductivity in non-simply connected domains introduced by Berger and Rubinstein in [15].

[3] see http://w3.ens-rennes.fr/math/simulations/MinimalPartitions/

4.1 Aharonov-Bohm effect

The Aharonov-Bohm effect [7] is one of the basic effects explained by quantum mechanics, but usually refers to an experiment related to scattering theory. According to Google Scholar, there is a huge literature in mathematics devoted to the analysis of this effect, starting with Ruisjenaars [81]. Another effect is related to bound states and gives in some sense a refined version of the diamagnetic effect. In the non-simply connected 2D-cases, when the magnetic field is identically 0, the circulations (modulo $2\pi\mathbb{Z}^d$) around each hole appear consequently as the unique relevant quantities. The limiting case when the holes are points will be our most important case. If we consider in Ω the Dirichlet realization $H_{\mathbf{A},V}$ of

$$\sum_j (D_{x_j} - A_j)^2 + V\,, \qquad \text{with } D_{x_j} = -i\partial_{x_j}\,,$$

the celebrated diamagnetic inequality due to Kato says:

$$\inf\ \sigma\left(H_{\mathbf{A},V}\right) \geq \inf\ \sigma\left(H_{\mathbf{0},V}\right)\ . \tag{15}$$

This inequality admits a kind of converse, showing its optimality (Lavine-O'Carroll-Helffer) (see the presentation in [48]).

Proposition 4.1.
Suppose that $\Omega \Subset \mathbb{R}^2$, $\mathbf{A} \in C^1(\overline{\Omega})$ and $V \in L^\infty(\Omega)$. Let $\lambda_{\mathbf{A},V}$ be the ground state of $H_{\mathbf{A},V}$. Then the following three properties are equivalent.

1. *$H_{\mathbf{A},V}$ and $H_{\mathbf{0},V}$ are unitarily equivalent;*
2. *$\lambda_{\mathbf{A},V} = \lambda_{\mathbf{0},V}$;*
3. *$\mathbf{A}$ satisfies the two conditions $\operatorname{curl}\mathbf{A} = 0$ and $\frac{1}{2\pi}\int_\gamma \mathbf{A} \in \mathbb{Z}$ on any closed path γ in Ω, where $\int_\gamma \mathbf{A}$ denotes the circulation of $\mathbf{A}$ along γ.*

4.2 Aharonov-Bohm operators

Let Ω be a planar domain and $\mathbf{p} = (p_1, p_2) \in \Omega$. Let us consider the Aharonov-Bohm Laplacian in a punctured domain $\dot{\Omega}_{\mathbf{p}} := \Omega \setminus \{\mathbf{p}\}$ with a singular magnetic potential and normalized flux α. We first introduce

$$\mathbf{A}^{\mathbf{p}}(\mathbf{x}) = (A_1^{\mathbf{p}}(\mathbf{x}), A_2^{\mathbf{p}}(\mathbf{x})) = \frac{(\mathbf{x}-\mathbf{p})^\perp}{|\mathbf{x}-\mathbf{p}|^2}, \qquad \text{with} \quad \mathbf{y}^\perp = (-y_2, y_1)\,.$$

This magnetic potential satisfies

$$\operatorname{curl}\mathbf{A}^{\mathbf{p}}(\mathbf{x}) = 0 \quad \text{ in } \dot{\Omega}_{\mathbf{p}}\,.$$

If $\mathbf{p} \in \Omega$, its circulation along a path of index 1 around $\mathbf{p}$ is 2π (or the flux created by $\mathbf{p}$). If $\mathbf{p} \notin \Omega$, $\mathbf{A}^{\mathbf{p}}$ is a gradient and the circulation along any path in Ω is zero. From now on, we renormalize the flux by dividing it by 2π.

The Aharonov-Bohm Hamiltonian with singularity $\mathbf{p}$ and flux α, written for brevity $H^{AB}(\dot{\Omega}_{\mathbf{p}}, \alpha)$, is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_{\mathbf{p}})$ and the differential operator

$$-\Delta_{\boldsymbol{\alpha}\mathbf{A}^{\mathbf{p}}} := (D_{x_1} - \alpha A_1^{\mathbf{p}})^2 + (D_{x_2} - \alpha A_2^{\mathbf{p}})^2 \,. \tag{16}$$

This construction can be extended to the case of a configuration with ℓ distinct points $\mathbf{p}_1, \ldots, \mathbf{p}_\ell$ (putting a flux α_j at each of these points). We just take as magnetic potential

$$\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{P}} = \sum_{j=1}^{\ell} \alpha_j \mathbf{A}^{\mathbf{p}_j} \,, \qquad \text{where} \quad \mathbf{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_\ell) \quad \text{and} \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell),$$

and consider the operator in $\dot{\Omega}_{\mathbf{P}} := \Omega \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_\ell\}$. Let us point out that the $\mathbf{p}_j$'s can be in $\mathbb{R}^2 \setminus \Omega$, and in particular in $\partial\Omega$. It is important to observe that if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ modulo $\mathbb{Z}^\ell$, then $H^{AB}(\dot{\Omega}_{\mathbf{P}}, \boldsymbol{\alpha})$ and $H^{AB}(\dot{\Omega}_{\mathbf{P}}, \boldsymbol{\alpha}')$ are unitary equivalent.

4.3 The case when the fluxes are $1/2$

Let us assume for the moment that there is a unique pole $\ell = 1$ and suppose that the flux α is $1/2$. For brevity, we omit α in the notation when it equals $1/2$. Let $K_{\mathbf{p}}$ be the antilinear operator

$$K_{\mathbf{p}} = \mathrm{e}^{i\theta_{\mathbf{p}}}\, \Gamma \,, \tag{17}$$

where Γ is the complex conjugation operator $\Gamma u = \bar{u}$ and $\theta_{\mathbf{p}}$ is such that $d\theta_{\mathbf{p}} = 2\mathbf{A}^{\mathbf{p}}$.
We note that, because the normalized flux of $2\mathbf{A}^{\mathbf{p}}$ belongs to $\mathbb{Z}$ for any path in $\dot{\Omega}_{\mathbf{p}}$, the function $\mathbf{x} \mapsto \exp i\theta_{\mathbf{p}}(\mathbf{x})$ is C^∞. A function u is called $K_{\mathbf{p}}$-real, if $K_{\mathbf{p}} u = u$. The operator $H^{AB}(\dot{\Omega}_{\mathbf{p}}) = H^{AB}(\dot{\Omega}_{\mathbf{p}}, \frac{1}{2})$ preserves the $K_{\mathbf{p}}$-real functions. Therefore we can consider a basis of $K_{\mathbf{p}}$-real eigenfunctions. Hence we only analyze the restriction of $H^{AB}(\dot{\Omega}_{\mathbf{p}})$ to the $K_{\mathbf{p}}$-real space $L^2_{K_{\mathbf{p}}}$ where

$$L^2_{K_{\mathbf{p}}}(\dot{\Omega}_{\mathbf{p}}) = \{u \in L^2(\dot{\Omega}_{\mathbf{p}}) \;:\; K_{\mathbf{p}}\, u = u\,\} \,.$$

If there are several poles ($\ell > 1$) and $\boldsymbol{\alpha} = (\frac{1}{2}, \ldots, \frac{1}{2})$, we can also construct the antilinear operator $K_{\mathbf{P}}$, where $\theta_{\mathbf{p}}$ in (17) is replaced by

$$\Theta_{\mathbf{P}} = \sum_{j=1}^{\ell} \theta_{\mathbf{p}_j} \,. \tag{18}$$

4.4 Nodal sets of $K_{\mathbf{P}}$-real eigenfunctions

As mentioned previously, under the half-integer flux condition, we can find a basis of $K_{\mathbf{P}}$-real eigenfunctions. It was shown in [56] and [8] that the $K_{\mathbf{P}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that, at each singular point $\mathbf{p}_j$ ($j = 1, \ldots, \ell$), an odd number $\nu(\mathbf{p}_j)$ of half-lines meet. So the only difference with the notion of regularity introduced for minimal partitions is that some $\nu(\mathbf{p}_j)$ can be equal to 1.
The ground state of $H^{AB}(\dot{\Omega}_{\mathbf{P}})$ has a very particular structure (see [15], [56] and [55]).

Proposition 4.2 (Slitting property)**.** *If $\mathcal{N}$ denotes the zero set of a $K_{\mathbf{P}}$-real eigenfunction of $H^{AB}(\dot{\Omega}_{\mathbf{P}})$ corresponding to the lowest eigenvalue, then $\overline{\Omega} \setminus \mathcal{N}$ is connected.*

This is illustrated in Figure 3. In particular, in the case of one pole, the proposition says that the zero set of a $K_{\mathbf{P}}$-real ground-state consists of a line joining the pole and the exterior boundary.

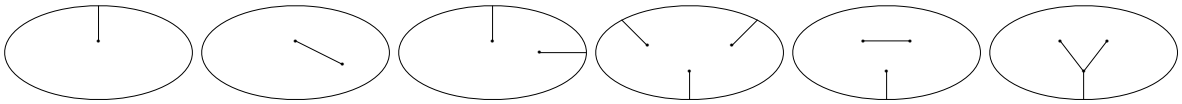


Figure 3: Possible topological types of nodal sets in function of the number ℓ of poles ($\ell = 1, 2, 3$).

It is also proven in [56]:

Proposition 4.3 (Multiplicity)**.** *The multiplicity m of the first eigenvalue satisfies*

$$m \leq \begin{cases} 2\,, & \text{for } \ell = 1, 2\,, \\ \ell\,, & \text{for } \ell \text{ odd, } \ell \geq 3\,, \\ \ell - 1\,, & \text{for } \ell \text{ even, } \ell \geq 4\,. \end{cases} \tag{19}$$

These two propositions are also true in the case of the Dirichlet realization of a Schrödinger operator in the form $H^{AB}(\dot{\Omega}_{\mathbf{P}}) + V$. They are actually also proved [55] for the Neumann problem and in the case of more general holes.

Coming back to more general eigenfunctions, we have:

Proposition 4.4. *The zero set of a $K_{\mathbf{P}}$-real eigenfunction of $H^{AB}(\dot{\Omega}_{\mathbf{P}})$ is the boundary set of a regular partition if and only if $\nu(\mathbf{p}_j) \geq 2$ for $j = 1, \ldots, \ell$.*

Let us illustrate the case of the square with one singular point. Figure 4 gives the nodal lines of some eigenfunctions of the Aharonov-Bohm operator: there are always one or three lines ending at the singular point (represented by a red dot). Note that only the fourth picture gives a regular and nice partition.

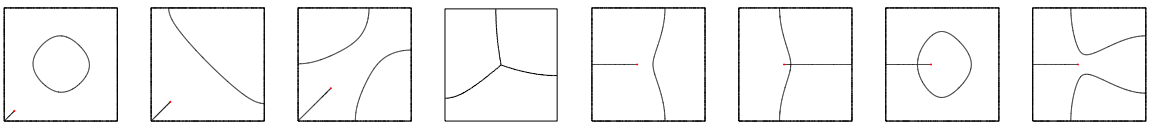


Figure 4: Nodal lines of some Aharonov-Bohm eigenfunctions on the square.

Our guess for the punctured square ($\mathbf{p}$ at the center) is that any nodal partition of a third $K_{\mathbf{p}}$-real eigenfunction gives a minimal 3-partition. Numerics shows that this is only true if the square is punctured at the center (see Figure 5 and [24] for a systematic study). Moreover the third eigenvalue is maximal there and has multiplicity two (see Figure 6).

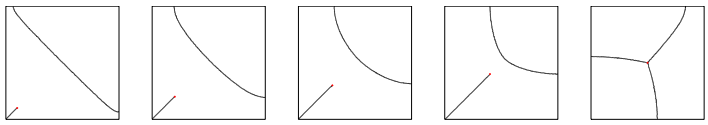


Figure 5: Nodal lines for the third Aharonov-Bohm eigenfunction in function of $\mathbf{p}$ on the diagonal.

4.5 Minimal partitions and Aharonov-Bohm operators

Helffer–Hoffmann-Ostenhof prove a magnetic characterization of minimal k-partitions (see [58, Theorem 5.1]):

Theorem 4.5. *Let Ω be simply connected and $\mathcal{D}$ be a minimal k-partition of Ω. Then $\mathcal{D}$ is the nodal partition of some k-th $K_{\mathbf{P}}$-real eigenfunction of $H^{AB}(\dot{\Omega}_{\mathbf{P}})$ with $\{\mathbf{p}_1, \dots, \mathbf{p}_\ell\} = X^{\mathsf{odd}}(\partial\mathcal{D})$.*

Proof. We return to the proof that a bipartite minimal partition is nodal for the Laplacian. Using the φ_j whose existence was recalled for minimal partitions, we can find a sequence $\varepsilon_j = \pm 1$ such that $\sum_j \varepsilon_j \exp(\frac{i}{2}\Theta_{\mathbf{P}}(\mathbf{x}))\, \varphi_j(\mathbf{x})$ is an eigenfunction of $H^{AB}(\dot{\Omega}_{\mathbf{P}})$, where $\Theta_{\mathbf{P}}$ was defined in (18). □

4.6 Continuity with respect to the poles

In the case of a unique singular point, [76], [30, Theorem 1.1] establish the continuity with respect to the singular point up to the boundary.

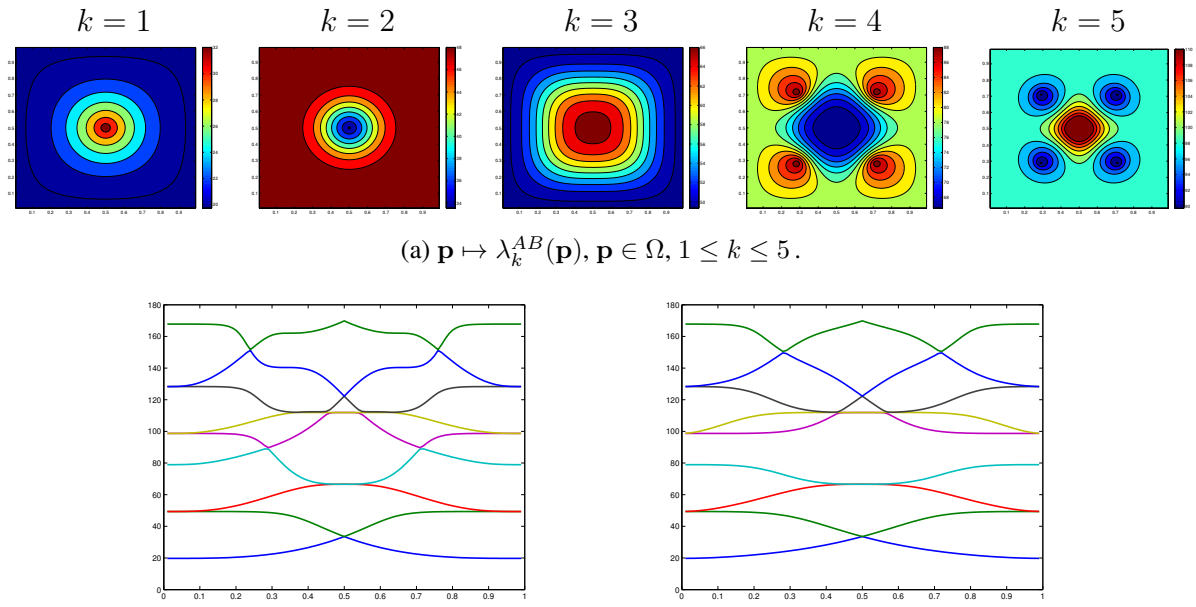
Theorem 4.6. *Let Ω be simply connected, $\alpha \in [0, 1)$, and let $\lambda_k^{AB}(\mathbf{p}, \alpha)$ be the k-th eigenvalue of $H^{AB}(\dot{\Omega}_{\mathbf{p}}, \alpha)$. Then the function $\mathbf{p} \in \Omega \mapsto \lambda_k^{AB}(\mathbf{p}, \alpha)$ admits a continuous extension to $\overline{\Omega}$ and*

$$\lim_{\mathbf{p} \to \partial\Omega} \lambda_k^{AB}(\mathbf{p}, \alpha) = \lambda_k(\Omega)\,, \qquad \forall k \geq 1\,. \tag{20}$$

The theorem implies that the function $\mathbf{p} \mapsto \lambda_k^{AB}(\mathbf{p}, \alpha)$ has an extremal point in $\overline{\Omega}$. Note also that $\lambda_k^{AB}(\mathbf{p}, \alpha)$ is well defined for $\mathbf{p} \notin \Omega$ and is equal to $\lambda_k(\Omega)$. One can indeed find a solution ϕ in Ω satisfying $d\phi = \mathbf{A}_{\mathbf{p}}$, and $u \mapsto \exp(i\alpha\phi)\, u$ defines the unitary transform intertwining $H(\Omega)$ and $H^{AB}(\dot{\Omega}_{\mathbf{p}}, \alpha)$. Figure 6a gives the first eigenvalues of $H^{AB}(\dot{\Omega}_{\mathbf{p}})$ in function of $\mathbf{p}$ in the square $\Omega = [0, 1]^2$ and demonstrates (20). When $\mathbf{p} = (1/2, 1/2)$, the eigenvalue is extremal and always double (see in particular Figures 6b and 6c which represent the first eigenvalues when the pole is either on a diagonal line or on a bisector line).

Let us analyze what can happen at an extremal point (see [76, Theorem 1.1], [30, Theorem 1.5]).

Theorem 4.7. *Suppose $\alpha = 1/2$. For any $k \geq 1$ and $\mathbf{p} \in \Omega$, we denote by $\varphi_k^{AB,\mathbf{p}}$ an eigenfunction associated with $\lambda_k^{AB}(\mathbf{p})$.*



(a) $\mathbf{p} \mapsto \lambda_k^{AB}(\mathbf{p})$, $\mathbf{p} \in \Omega$, $1 \leq k \leq 5$.

(b) $p \mapsto \lambda_k^{AB}(\mathbf{p})$ with $\mathbf{p} = (p,p)$, $1 \leq k \leq 9$. (c) $p \mapsto \lambda_k^{AB}(\mathbf{p})$ with $\mathbf{p} = (p, \frac{1}{2})$, $1 \leq k \leq 9$.

Figure 6: Aharonov-Bohm eigenvalues $\lambda_k^{AB}(\mathbf{p})$ on the square as functions of the pole $\mathbf{p}$.

– *If $\varphi_k^{AB,\mathbf{p}}$ has a zero of order $1/2$ at $\mathbf{p} \in \Omega$, then either $\lambda_k^{AB}(\mathbf{p})$ has multiplicity at least 2, or $\mathbf{p}$ is not an extremal point of the map $\mathbf{x} \mapsto \lambda_k^{AB}(\mathbf{x})$.*
– *If $\mathbf{p} \in \Omega$ is an extremal point of $\mathbf{x} \mapsto \lambda_k^{AB}(\mathbf{x})$, then either $\lambda_k^{AB}(\mathbf{p})$ has multiplicity at least 2, or $\varphi_k^{AB,\mathbf{p}}$ has a zero of order $m/2$ at $\mathbf{p}$, $m \geq 3$ odd.*

This theorem gives an interesting necessary condition for candidates to be minimal partitions. Indeed, knowing the behavior of the eigenvalues of the Aharonov-Bohm operator, we can localize the position of the critical point for which the associated eigenfunction can produce a nice partition (with singular point where an odd number of lines end). We observe in Figure 6 that the eigenvalue is never simple at an extremal point.

When there are several poles, the continuity result of Theorem 4.6 still holds. We will briefly address this result (see [72] for the proof and more details). This is rather clear in $\Omega^\ell \setminus \mathcal{C}$, where $\mathcal{C}$ denotes the $\mathbf{P}$'s such that $\mathbf{p}_i \neq \mathbf{p}_j$ when $i \neq j$. It is then convenient to extend the function $\mathbf{P} \mapsto \lambda_k^{AB}(\mathbf{P}, \boldsymbol{\alpha})$ to $(\mathbb{R}^2)^\ell$. We define $\lambda_k^{AB}(\mathbf{P}, \boldsymbol{\alpha})$ as the k-th eigenvalue of $H^{AB}(\dot{\Omega}_{\tilde{\mathbf{P}}}, \tilde{\boldsymbol{\alpha}})$, where the m-tuple $\tilde{\mathbf{P}} = (\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_m)$ contains once, and only once, each point appearing in $\mathbf{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_\ell)$ and where $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_M)$ with $\tilde{\alpha}_k = \sum_{j,\, \mathbf{p}_j = \tilde{\mathbf{p}}_k} \alpha_j$, for $1 \leq k \leq m$.

Theorem 4.8. *If $k \geq 1$ and $\boldsymbol{\alpha} \in \mathbb{R}^\ell$, then the function $\mathbf{P} \mapsto \lambda_k^{AB}(\mathbf{P}, \boldsymbol{\alpha})$ is continuous in $\mathbb{R}^{2\ell}$.*

This result generalizes Theorems 4.6 and 4.7. It implies in particular continuity of the eigenvalues when one point tends to $\partial\Omega$, or in the case of coalescing points. For example, take $\ell = 2$, $\alpha_1 = \alpha_2 = 1/2$, $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2)$ and suppose that $\mathbf{p}_1$ and $\mathbf{p}_2$ tend to some $\mathbf{p}$ in Ω. One obtains in this case that $\lambda_k^{AB}(\mathbf{P}, \boldsymbol{\alpha})$ tends to $\lambda_k(\Omega)$.

Remark 4.9. *More results on the Aharonov-Bohm eigenvalues as function of the poles can be found in [24, 76, 30, 1, 72, 4, 2, 5, 3, 6]. We have emphasized in this section only the results which have direct applications to the research of candidates for minimal partitions.*
In many of the papers analyzing minimal partitions, the authors refer to a double covering argument. Although this point of view (which appears first in [56] in the case of domains with holes) is essentially equivalent to the Aharonov approach, it has a more geometrical flavor.

5 On the asymptotic behavior of minimal k-partitions

The hexagon has fascinating properties and appears naturally in many contexts (for example the honeycomb). Let $\hexagon$ be a regular hexagon of unit area. If we consider polygons generating a tiling, the ground state energy $\lambda(\hexagon)$ gives the smallest value (at least in comparison with the square, the rectangle and the equilateral triangle). In this section we analyze the asymptotic behavior of minimal k-partitions as $k \to +\infty$.

5.1 The hexagonal conjecture

Conjecture 5.1. *The limit of $\mathfrak{L}_k(\Omega)/k$ as $k \to +\infty$ exists and*

$$|\Omega| \lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(\hexagon) \ .$$

Similarly, one has

Conjecture 5.2. *The limit of $\mathfrak{L}_{k,1}(\Omega)/k$ as $k \to +\infty$ exists and*

$$|\Omega| \lim_{k\to+\infty} \frac{\mathfrak{L}_{k,1}(\Omega)}{k} = \lambda(\hexagon) \ . \tag{21}$$

These conjectures, that we learned from M. Van den Berg in 2006, are also mentioned in Caffarelli-Lin [37] for $\mathfrak{L}_{k,1}$ and imply that the limit is independent of Ω. Of course the optimality of the regular hexagonal tiling appears in various contexts in physics. By keeping the hexagons belonging to the intersection of Ω with the hexagonal tiling and using the monotonicity of $\mathfrak{L}_k$ for the inclusion, it is easy to show the upper bound in Conjecture 5.1,

$$|\Omega| \limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \lambda(\hexagon) \ . \tag{22}$$

We recall that the Faber-Krahn inequality (4) gives a weaker lower bound,

$$|\Omega| \frac{\mathfrak{L}_k(\Omega)}{k} \geq |\Omega| \frac{\mathfrak{L}_{k,1}(\Omega)}{k} \geq \lambda(\bigcirc) \ . \tag{23}$$

Note that Bourgain [32] and Steinerberger [83] have recently improved the lower bound by using an improved Faber-Krahn inequality together with considerations on packing property by disks.
The inequality $\mathfrak{L}_{k,1}(\Omega) \leq \mathfrak{L}_k(\Omega)$ together with the upper bound (22) shows that the second

conjecture implies the first one. Conjecture 5.1 has been explored in [27] by checking numerically nontrivial consequences of this conjecture. Other recent numerical computations devoted to $\lim_{k\to+\infty}\frac{1}{k}\mathfrak{L}_{k,1}(\Omega)$ and to the asymptotic structure of the minimal partitions are given by Bourdin-Bucur-Oudet [31].

The hexagonal conjecture leads to a natural corresponding hexagonal conjecture for the length of the boundary set, namely

Conjecture 5.3. *Let $\ell(\hexagon) = 2\sqrt{2\sqrt{3}}$ be the length of the boundary of $\hexagon$. Then*

$$\lim_{k\to+\infty}\frac{|\partial\mathcal{D}_k|}{\sqrt{k}} = \frac{1}{2}\ell(\hexagon)\sqrt{|\Omega|}\,. \tag{24}$$

This point is discussed in [11] in connection with the celebrated theorem of Hales [51] proving the honeycomb conjecture.

5.2 Lower bound for the number of singular points

It has been established since 1925 (A. Stern, H. Lewy, J. Leydold, Bérard-Helffer; see [12] and references therein), that there are domains for which there exist an infinite sequence of eigenvalues of the Laplacian for which the corresponding eigenvalues have a fixed number of nodal domains and critical points in the zero set. The next result (see [61], [54]) shows that the situation is quite different for minimal partitions.

Theorem 5.4. *For any sequence $(\mathcal{D}_k)_{k\in\mathbb{N}}$ of regular minimal k-partitions, we have*

$$\liminf_{k\to\infty}\frac{\sharp X^{\mathsf{odd}}(\partial\mathcal{D}_k)}{k} > 0\,. \tag{25}$$

Although inspired by the proof of Pleijel's theorem, this proof includes (for any k) a lower bound in Weyl's formula for the eigenvalue $\mathfrak{L}_k$ of the Aharonov-Bohm operator $H^{AB}(\dot{\Omega}_{\mathbf{P}})$ associated with the odd singular points of $\mathcal{D}_k$. The proof gives an explicit but very small lower bound in (25) which is independent of the sequence. This is to compare with the upper bound (14), which gives

$$\limsup_{k\to\infty}\frac{\sharp X^{\mathsf{odd}}(\partial\mathcal{D}_k)}{k} \le 2\,.$$

Remark 5.5. *The hexagonal conjecture in the case of a compact Riemannian manifold is the same. We refer to [11] for the details, the idea being that, for k large, the local structure of the manifold plays the main role, like for Pleijel's formula (see [14]). In [46] the authors analyze numerically the validity of the hexagonal conjecture in the case of the sphere for $\mathfrak{L}_{k,1}$. Using Euler's formula, one can conjecture that there are $(k-12)$ hexagons and 12 pentagons for k large enough. In the case of a planar domain one expects curvilinear hexagons inside Ω, around $\sqrt{k}$ pentagons close to the boundary (see [31]) and a few other polygons. Let us mention also the recent related results of D. Bucur and collaborators [34, 35].*

6 Open problems

We finish this survey by recalling other open problems.

6.1 Problems related to Pleijel's theorem

Let us come back to the strong version of Pleijel's theorem (see Subsection 2.3). The relation of the number of nodal domains and the properties of the Pleijel constant

$$Pl(\Omega) := \limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n},$$

appearing in (7), is still very mysterious. Here $\Omega \subset \mathbb{R}^d$ is a bounded domain and we consider a Dirichlet Laplacian. But, as observed in Remark 2.6, one could also consider the analogous problems for Schrödinger operators. We mention some natural questions - some of them might seem rather ridiculous, but we just want to demonstrate how little is understood.

1. Find some bounded domains $\Omega \subset \mathbb{R}^d$ so that one can work out the Pleijel constant $Pl(\Omega)$ explicitly. For rectangles $\mathcal{R}(a,b)$ with sidelengths a, b with $\frac{a^2}{b^2}$ irrational it is known that $Pl(\Omega) = \frac{2}{\pi}$, see *e.g.* [61]. Based on numerical work [21], Polterovich [79] moreover conjectured that $Pl(\Omega) \leq \frac{2}{\pi}$. For the harmonic oscillator $H^{\mathsf{osc}} = -\Delta + \sum_{j=1}^d a_j^2 x_j^2$ on $\mathbb{R}^d$ with $a_1, \ldots, a_d$ rationally independent, Charron [38] showed that $\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} = \frac{d!}{d^d}$.

2. It is not known whether $Pl(\Omega) > 0$ always holds and even whether $\limsup_{n\to\infty} \mu(\varphi_n) = +\infty$ in general. Take a bounded domain $\Omega \subset \mathbb{R}^2$. If we consider an example where we can work out the eigenvalues λ_n and the associated eigenfunctions φ_n explicitly, then we have always that $Pl(\Omega) > 0$. In general almost nothing is known. There are some very subtle families of manifolds for which it has been shown that $\limsup_{n\to\infty} \mu(\varphi_n) = +\infty$, see [49, 50, 70, 87]. But for membranes this question is wide open.
   Here is a simple problem:
   Prove or disprove that there exists an integer K such that, for all bounded (perhaps simply connected) domains, there exists an eigenfunction φ_k associated with λ_k with $k \leq K$ and $\mu(\varphi_k) \geq 3$.
   A related problem is the following:
   Find $\Omega \subset \mathbb{R}^2$ so that $\mu(\varphi_k) = 2$ for $1 < k \leq 5$.
   It is not at all clear that such a membrane exists. There are examples where λ_2 has multiplicity 3, in the case of the sphere $\mathbb{S}^2$ for instance, but also for not simply connected domains in $\mathbb{R}^2$ (see the paper by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [67] on the nodal line conjecture).

   For the higher dimensional case, Colin de Verdière [41] has shown that one can have arbitrarily high multiplicity of the second eigenvalue for certain Riemannian eigenvalue problems.

3. Take any Ω (with Dirichlet or Neumann boundary condition) and consider

$$\mathfrak{N}(\Omega) = \{k \in \mathbb{N} \ : \ \exists \text{ an eigenfunction } \varphi \text{ with } \mu(\varphi) = k\}. \tag{26}$$

   For problems where one can work out the eigenvalues and the corresponding eigenfunctions explicitly, we have usually $\mathfrak{N}(\Omega) = \mathbb{N}$. For the circle $\mathbb{S}^1$ we have $\mathfrak{N} = \{1\} \cup 2\mathbb{N}$.

This is also the case for the torus $T(a,b)$ where $T(a,b)$ is the rectangle $\mathcal{R}(a,b)$ with periodic boundary condition and with $(a/b)^2$ irrational. If this assumption on a,b does not hold, then there are examples of $\mu(\varphi)=3$, see [73]. Here comes a problem:

Prove or disprove that for bounded $\Omega \subset \mathbb{R}^2$ either $\mathfrak{N}=\mathbb{N}$ or $\mathfrak{N}=\{1\}\cup 2\mathbb{N}$.

For instance can it happen that there is a domain so that no eigenfunction has four nodal domains? Analogous questions can be also asked for the d-dimensional case, Schrödinger operators and for Laplace Beltrami operators on bounded manifolds.

6.2 Open problems on minimal partitions

We do not come back to the Mercedes star conjecture for the disk, which was discussed in Subsection 3.6 but there are related problems for which we have natural guesses for minimal 3-partitions and, more generally, minimal k-partitions. Consider the equilateral triangle and the regular hexagon. In both cases the boundary set of the 3-partition should consist of three straight segments which start in the middle of the side and meet at the center with the angle $2\pi/3$. For the regular hexagon it is similar, the three straight segments start from the middle of three sides which do not neighbor each other and meet in the center of the hexagon. A related natural guess is available for the non-nodal minimal spectral 5-partition for the disk. There one expects that the minimal partition is created by five segments which start form the origin and meet there with angle $2\pi/5$ (see Figure 1b). For all those examples there is strong numerical evidence; see [22, 23]. We now mention most of the cases of non-nodal minimal spectral partitions for which the topology of minimal k-partition is known. The simplest case is $\mathbb{S}^1$. Here we know everything: all minimal $2k$-partitions are nodal and the minimal $(2k+1)$-partitions just are given by $(0,\frac{2\pi}{2k+1})$ and its rotations by multiples of $\frac{2\pi}{2k+1}$. To see this just go to the double covering and look at $(0,4\pi)$ with periodic boundary conditions, or consider $(D_x-\frac{1}{2})^2$ on $(0,2\pi)$ with periodic condition.

We consider the Laplacian operator on the rectangle $\mathcal{R}(1,b)=(0,1)\times(0,b)$. Two cases will be mentioned.

First we take periodic boundary conditions on the interval (0,1) by identifying $x_1=0$ with $x_1=1$. For $x_2=0$ and $x_2=b$ we take Neumann boundary conditions. The spectrum of this operator $H^{\mathsf{per},N}$ is

$$\sigma(H^{\mathsf{per},N})=\Big\{\pi^2(4m^2+\frac{n^2}{b^2}),(m,n)\in\mathbb{Z}\times\mathbb{N}\Big\}.$$

Since $\lambda_1=0$ and $\lambda_2=\lambda_3$ for $b<1/2$, $\mathfrak{L}_3$ is associated to a non-nodal partition. If we let $b>0$ sufficiently small we are "near" the case of $\mathbb{S}^1$. One can then go to the double covering as for the circle and consider the eigenvalues. For $b\le(2\sqrt{5})^{-1}$, the minimal 3-partition is then given by $D_1=(0,1/3)\times(0,b),\ D_2=(1/3,2/3)\times(0,b), D_3=(2/3,1)\times(0,b)$, see [59]. It was only possible with Neumann boundary conditions. For Dirichlet one would expect also such a minimal partition. For larger odd k one gets similar results (requiring smaller b). Hence the problem is:

Prove such a result with Dirichlet boundary conditions and improve the conditions on the $b's$.

Next we consider the torus [60, 29]. Equivalently we consider periodic boundary conditions on the intervals $(0,1)$ and $(0,b)$. Again one can expect, for small b, a situation "near" to the circle. Things are more involved but again one can reduce the problem to spectral problems on suitable coverings. There are many nice problems coming up here, which are discussed in [29].

Finally we come back to $\mathbb{S}^2$ (see Section 3.5). Since only $\lambda_1(\mathbb{S}^2)$ and $\lambda_2(\mathbb{S}^2)$ are Courant-sharp, any $\mathfrak{L}_k(\mathbb{S}^2)$ corresponds to a non-nodal minimal partition for $k > 2$. In [64] it was shown (see Figure 7) that, up to rotation, the minimal 3-partition is unique and that its boundary set consists of three half great circles which connect the north-pole and the south-pole and meet each other with angles $2\pi/3$. There is an open question in harmonic analysis known as the

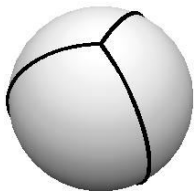


Figure 7: Minimal 3-partition of the sphere.

Bishop conjecture (solved for 1-minimal 2-partitions) [19]:
Show that the minimal 3*-partition of* $\mathbb{S}^2$ *is a* 1*-minimal* 3*-partition.*

In [64] possible candidates for minimal 4-partitions were discussed and one natural guess is the regular tetrahedron. More generally, one might ask the question whether platonic solids (corners on the sphere connected by segments of great circles) are possible candidates. First we note that the octahedron would lead to a bipartite partition hence the minimal 8-partition must be something else. But the tetrahedron, the cube, the dodecahedron and the icosahedron would be possible candidates. There is a related isoperimetric problem whose relation to spectral minimal partitions might be interesting. The problem is to find the least perimeter partition of $\mathbb{S}^2$ into k regions of equal area. Up to now the cases $k = 3, 4, 12$, [75, 47, 52], have been solved. For $k = 3$ the perimeter partition is also the spectral minimal 3-partition. We can then propose the following problem:
Prove or disprove that for $k = 4, 6, 12, 20$ *(each case would be interesting) that the minimal k-partitions correspond to the platonic solids.*

Remark 6.1. *Let us also mention that if the nodal line conjecture holds (see [67] and references therein), say for simply connected domains, this would imply a lot about the geometry of minimal spectral partitions.*

Finally, we want to mention some very interesting recent results by Smilansky and collaborators, [17] for the membrane case in $\Omega \subset \mathbb{R}^d$ and [9] for the discrete case. We just indicate their results. They consider the generic case, simple eigenvalues and eigenfunctions which have no higher order zeros, and investigate for the k-th eigenfunction φ_k the *nodal deficiency* $d_k = k - \mu(\varphi_k)$. They define a functional, somehow a bit in the spirit of Equation (2), and show

that the critical point of this functional corresponds to a nodal partition. Moreover the Morse index of the critical point turns out to be the deficiency index. It would be very interesting to investigate whether this approach can be extended to the non-generic case.

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Ludwig D. Faddeev

(March 23, 1934 – February 26, 2017)

On February 26, 2017, after a long fight with cancer Ludwig Dmitrievich Faddeev passed away. For the authors of this article Faddeev was a teacher and a constant source of scientific inspiration for many formative years.

First steps

Ludwig Faddeev was born in Leningrad (now St. Petersburg), USSR (now Russian Federation) on March 23, 1934, to a family of mathematicians Dmitrii Konstantinovich Faddeev[4] and Vera Nikolaevna Faddeeva[5]. From childhood Ludwig was exposed to classical music, to piano in particular, and he seriously considered attending the Leningrad Conservatory as an alternative to the university.

D. K. Faddeev playing piano.

Ludwig playing piano.

At the time of his graduation from high school in 1951 his father was the Dean of the Mathematics Department at the Leningrad State University, and so young Ludwig decided to go to the Physics Department. When he was a junior, a new Chair [6] of Mathematical Physics was established at the Physics Department. The first graduating class in mathematical physics had five students, two young men and three young women, among whom were L. D. Faddeev and N. Uraltseva.

At the time O. A. Ladyzhenskaya was a young energetic professor at the department and had great respect among students. She was lecturing on the theory of functions of complex variables, operator algebras, partial differential equations, and other subjects. When Ludwig was in the fourth year of his studies, she organized a student seminar based on the book by K. O. Friedrichs, "Mathematical aspects of the quantum theory of fields" [1]. Ladyzhenskaya had learned about this book from her colleagues at Courant. Faddeev was a key participant in the seminar. The book was translated and studied in every detail. According to his recollections, Faddeev was particularly impressed by the last chapter (the appendix) of the book, which

[4] D. K. Faddeev was one of the leading figures in algebra in the Soviet Union. Among his many results was the invention of group cohomology. This work was done while evacuated from Leningrad, mostly in Kazan, during the Siege of Leningrad (Leningrad Blockade) by the German Army Group North during World War II, when Ludwig was 7 to 9 years old. Due to the unfortunate timing this work was noticed only later, after the works of Eilenberg and MacLane.

[5] V. N. Faddeeva (maiden Zamyatina) was an applied mathematician working on numerical methods. Unfortunately, very few photos of her exist, and we did not find any that would be representative.

[6] Administratively the Physics Department (as any other department at any other university in the USSR) consisted of more specialized groups (called Kafedras, or Chairs, or Branches). Typically, each Chair was an umbrella unit for several professors.

was focused on scattering theory. He successfully used Friedrichs's perturbation theory of the continuum spectrum many times. It was an important tool in his study of scattering and inverse problems. At this time Ladyzhenskaya became his adviser. She also suggested to him to study the work of N. Levinson [2] on inverse scattering theory and to present it at the seminar. This was a very profound moment: inverse scattering problems influenced Ludwig's research for very long time. Regarding these times he wrote about O. A. Ladyzhenskaya: "I am forever grateful for the direction she gave me..."

O. A. Ladyzhenskaya.

He graduated from the University with the equivalent of a master degree in 1956 and continued to graduate school. At the time quantum mechanics and quantum theory were still a relatively young subject. Indeed, from the time when the Schödinger equation appeared in 1926 only thirty years had passed.

By that time important results in understanding the radial part of the three-dimensional Schrödinger equation had started to appear. In particular various aspects of the inverse problem for the s-channel of the Schrödinger equation, i.e. the Sturm-Liouville problem on a half line, were solved by Gelfand and Levitan [6], Marchenko [7] and Krein [8]. In 1958, at the suggestion of Ladyzhenskaya, N. N. Bogolyubov invited Faddeev as a speaker at the opening conference of the Theoretical Physics Laboratory in Dubna to present a survey and recent results on the one-dimensional Schrödinger equation. In the audience were I. M. Gelfand, B. M. Levitan, M. G. Krein,V. A. Marchenko, and other prominent mathematicians and physicists working in this field. The talk was very well received by the experts and Ludwig was invited to write a survey on the subject for the *Uspekhi Mathematicheskikh Nauk* (the central mathematical journal in the country) [22] where, among other things, he demonstrated the equivalence of the approaches by Gelfand-Levitan, Marchenko and Krein. This survey became a handbook

for generations of people working on this subject. It is interesting that Ludwig's presentation at this conference initially was prepared as the report for his PhD qualifying exam.

Ludwig defended his PhD thesis in 1959. In his dissertation he solved the inverse scattering problem for the Schödinger equation on the line with a rapidly decaying potential and found dispersion relations for scattering amplitudes[7].

The three-body problem and the quantum inverse scattering problem in three dimensions

After defending his PhD and settling as a researcher at LOMI[8] he focused on two problems: inverse scattering for a three-dimensional Schrödinger operator and the three-body scattering problem[9]. At the time he was reading a lot, especially on scattering theory. In particular, he studied the approaches by Lippman and Schwinger, Gell-Mann and Goldberger, Epstein and others. There was also the work by Skornyakov and Ter-Martirosyan, where the three-particle scattering problem was solved for δ-potentials.

Gradually, the outlines of Faddeev's work on three-particle scattering started to emerge. One of the technical tools important for his work is the reconstruction (regularization) of integral kernels. As he recalled, the idea of regularization came to him in 1960. His study of the Thirring model and of Epstein's paper [9] were important steps to come up with this idea. He discussed it with Gribov, which gave him confidence that the approach was new to physicists. The complete solution of the three-particle problem with rapidly decaying potential was published in a series of papers in 1960-1963. In 1962, Faddeev gave a sectional talk on the solution of three-particle scattering at the International Congress of Mathematicians in Stockholm, where among other participants from the Soviet Union were Gelfand, Kolmogorov, Linnik, Novikov, Patetsky-Shapiro, and Shafarevich. As he recalled his result went largely unnoticed. Ludwig's work on three-particle scattering was truly appreciated only after a paper by C. Lovelace [10], in which practical aspects of Faddeev's equations were developed. Immediately afterwards it was widely recognized as a milestone. Now it is commemorated by the Faddeev's medal [11], which is awarded for the best work on multi-particle scattering.

In 1963 Faddeev defended his habilitation degree. His dissertation was based on his work on three-particle scattering. The defense was in Moscow at the Stekov Mathematical Institute. His referees ("opponents" in Russian) were I. M. Gelfand, A. Ya. Povzner and V. S. Vladimirov.

The inverse problem for the three-dimensional Schrödinger operator was solved in a series of papers [12] where he described necessary and sufficient conditions for reconstructing the potential from the scattering data. After he completed this work he never returned to the analysis of the Schrödinger operator. Several of his students and colleagues completed various aspects of the program he outlined and extended the scope of the results, most notably V. S. Buslaev, D. R. Yafaev, S. P. Merkuriev, O. A. Yakubovsky.

[7] Landau and Lifshitz in their famous textbook "Quantum mechanics" (chapter XVII, section 130) outlined the proof of the dispersion relation for forward scattering and noted: "The idea of the proof belongs to L. D. Faddeev (1958)."

[8] LOMI is the abbreviation of Russian Leningradskoe Otdelenie Matematichesogo Instituta, or in English translation, Leningrad Branch of the Mathematical Institute (of the Steklov Institute of the Academy of Sciences).

[9] He recalled this period in an interview [4].

Yang-Mills and Faddeev-Popov ghosts

One of the main goals of Ludwig, after developing the theory of three-particle scattering, was to understand the quantum theory of fields (the preferred name is now *quantum field theory*) and apply it to gravity. Quantization of gravity was his main goal at this time. He did a lot of reading, studying classical papers of Feynman and Schwinger. One of his favorite books at the time was Lichnerovich's *Théories rélativistes de la gravitation et de l'électromagnétisme. Rélativité générale et théories unitaires* [13].

During this time a problem in quantum electrodynamics, known as the Landau pole, was discovered by Landau and Pomeranchuk. This cast doubts on concepts of local quantum field theory. Just before his tragic accident Landau wrote an influential article [15] in which he said[10]: "We are driven to the conclusion that the Hamiltonian method for strong interaction is dead and must be buried, although of course with deserved honor." In 1961, Faddeev published a joint paper with Berezin where he studied the renormalization of the wave function of the Schrödinger operator with zero radius [16]. One of their conclusions was that renormalization is not necessarily an artifact of perturbation theory, but that it can be approached by different methods. In their paper the non-perturbative method was the theory of self-adjoint extensions. This was one of the reasons for Ludwig to continue with Hamiltonian field theory despite widespread belief at the time that this path would lead to a dead end.

In 1964, a collection of translated papers, including the original paper by Yang and Mills, was published in Russian [17]. It immediately caught Ludwig's attention. By chance at the time Ludwig came across another book by Lichnerovich, on connections [14]. He immediately recognized that Yang-Mills fields are connections, and decided to focus on Yang-Mills theory first instead of more complicated Einstein gravity. During this time he already started collaborating with V.N. Popov. Soon this collaboration resulted in the famous article [18] setting out the correct perturbation theory (Feynman rules) for quantum Yang-Mills theory.

The main observation there was that because the transformation laws of non-abelian connections are nonlinear, the Jacobian of the mapping between gauge orbits and the gauge-fixing local section of the gauge group action needs to be taken into account. In perturbation theory this can be realized by the introduction of a non-physical fermionic "ghost" field, which became known as Faddeev-Popov ghost.

This revolutionary observation had great impact on the Yang-Mills theory. It defined a working perturbative expansion in terms of Feynman diagrams. The next step was the proof of the renormalizability of the Yang-Mills theory and the discovery that it is asymptotically free, i.e. free of the Landau's zero-charge paradox. Ultimately, it lead to the construction of Hamiltonian quantum field theory (the Standard Model) unifying all interactions except gravity. All computations in these developments were done using the Faddeev-Popov technique, for details see [20].

[10]This period of developments in quantum field theory is described clearly and concisely in the Nobel lecture by D. Gross [20].

Soliton equations as integrable systems

In 1971, Ludwig gave a talk at a symposium in Novosibirsk on his work on inverse scattering problem for three-dimensional Schrödinger equation. At the time V. E. Zakharov was working in Novosibirsk and after Faddeev's talk he explained to him the remarkable paper by Gardener, Green, Kruskal, and Miura on the method for solving the KdV equation and about Lax's interpretation of this work. This discussion led to the joint paper [23] by Faddeev and Zakharov, where they proved that the KdV equation is an infinite-dimensional completely integrable Hamiltonian system.

Finite-dimensional completely integrable Hamiltonian systems have their origin in classical papers by Euler, Lagrange, Jacobi, and Kovalevskaya on the rotation of a rigid body. But by the middle of the 20th century this subject was almost dead: there were no new finite-dimensional examples and not a single infinite-dimensional nonlinear example. In [23] Faddeev and Zakharov gave a very interesting infinite-dimensional example and laid the foundation for a new class of completely integrable systems which can be solved by the inverse-scattering method. In this sense it was a result of fundamental importance.

During his trip to the US in 1972, Faddeev gave a talk on his work with Zakharov. J. Klauder, who attended this talk, explained to Faddeev that there is another important equation which is relativistically invariant, the sine-Gordon equation[11]. It captured Faddeev's attention as a candidate for a classical relativistic nonlinear integrable field theory and in 1973 he started the search for a Lax pair with his new student L. Takhtajan. Remarkably, such a Lax operator was found [26][12]. In the subsequent works [28] and [29] the complete theory of the sine-Gordon equation as an infinite-dimensional completely integrable Hamiltonian system with a topological charge was developed.

The study of infinite-dimensional completely integrable systems was continued by Faddeev's students in the Laboratory of Mathematical Methods of Theoretical Physics in LOMI. Among them were P. P. Kulish, A. G. Reyman, N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky, E. K. Sklyanin and L. A. Takhtajan. Their body of work is summarized in the monograph [30].

These results built foundations for the theory of Poisson-Lie groups and Lie bialgebras later developed by V.G. Drinfeld [31]. This framework, together with Kostant's ideas and with [32], was developed in [33] into a scheme which produces integrable systems out of Poisson-Lie groups and solves their dynamics in terms of the factorization of Poisson-Lie groups.

Quantization of solitons

From the very beginning of the study of soliton equations Faddeev had in mind their potential for quantization, for constructing non-perturbative two-dimensional quantum field theories. From this point of view the sine-Gordon equation, being a relativistic invariant, was especially interesting. Faddeev correctly anticipated that the quantization of this infinite-dimensional Hamiltonian completely integrable system would be a relativistic quantum theory with rich

[11] This equation is a classical differential equation which describes embeddings of surfaces with constant negative curvature in $\mathbb{R}^3$. It also appears in nonlinear optics and superconductivity.

[12] It was also discovered independently in [27].

mass spectrum, where particles correspond to soliton solutions to the classical sine-Gordon equation. The first step in this direction was the semiclassical quantization with the hope that the quantum system would be an integrable local quantum field theory.

In 1974, Kulish demonstrated that in the semiclassical framework the SG model has infinitely many local conservation laws, which implies that the scattering must be purely elastic. This unexpected result was proven in perturbation theory by Arefieva and Korepin. This result, together with the locality of integrals of motion led to the conclusion that the multiparticle scattering in such systems reduces to a sequence of two particle scattering. Similar results were obtained in the paper by Faddeev, Kulish and Manakov [34] for the nonlinear Schrödinger equation which can be regarded as a non-relativistic limit of SG. The account of the semiclassical quantization of solitons is given in the review article by Faddeev and Korepin in [37].

Factorized scattering appeared earlier, in the pioneering work by C.N.Yang [35], who constructed eigenvectors and eigenvalues of the Hamiltonian of one-dimensional non-identical point particles with δ-function interactions. In this paper the Yang-Baxter equation first appeared, but in this case the scattering was non-relativistic.

The idea of factorized scattering in the relativistic setting was immediately picked up by physicists, and the wonderful paper by Zamolodchikov and Zamolodchikov [36] appeared immediately after the first convincing results on quantization of solitons.

From quantum integrable systems to quantum groups

Now we come to the Baxter part of the Yang-Baxter equation. Motivated by works on the Bethe ansatz and diagonalization of Hamiltonians of spin chains, Faddeev and several people in his Lab, including Kulish, Sklyanin and Takhtajan, started to study Baxter's papers [40] on the solution of the 8-vertex model in the late 70s [13]. At first the linear-algebra manipulations involved in [39] seemed far from what he had been doing before. Then a beautiful picture emerged where the inverse-transform method used in the theory of solitons merged with the core element of Baxter's work, and as a result a deeper understanding of what Baxter did and a method for quantizing Hamiltonian soliton equations emerged. In particular, eigenfunctions of the transfer-matrix of the 6-vertex, which had earlier been constructed by Lieb using the coordinate Bethe ansatz, now got a new algebraic interpretation through the "algebraic Bethe ansatz" [43, 45].

The first examples were the nonlinear Schrödinger model [42] and the sine-Gordon model [43]. Eigenstates for the quantum sine-Gordon model were constructed in [43] using an approximation consistent with a ultra-violet regularization of the model. The exact ultraviolet regularization was later constructed in [44].

On these two examples a general method for constructing quantum integrable systems emerged and became known as the quantum inverse transform method, formulated in [42, 43, 45]. It is based on solutions to the Yang-Baxter equation[14].

[13]The book by Baxter [39] was translated into Russian in 1985.

[14]The term "Baxter-Yang" equation was introduced in [45] (in alphabetical order); it later became "Yang-Baxter" equation (in chronological order).

L. D. Faddeev, C. N. Yang and R. J. Baxter.

This also gave a new intuition for the classical Hamiltonian structures for completely integrable systems, when Sklyanin introduced the notion of classical r-matrix. In 1982 Faddeev gave very influential lectures at the Les Houches summer school in Theoretical Physics on quantum integrable systems, solutions to the Yang-Baxter equation, and the Bethe ansatz. A number of other important models were solved, for example in [46] new integrable spin chains were discovered, etc.

It was a very exciting time: Faddeev's Lab was bustling with new ideas and with new results, and with many interesting people visiting from other institutions in the Soviet Union and from abroad. The method of classical r-matrix evolved into the factorization scheme for solving soliton equations [33] and eventually to the notions of Lie bialgebras and Poisson-Lie groups [31]. The quantum inverse transform method, involving solutions to the Yang-Baxter equation, gave rise to the theory of quantum groups [31] and to a new chapter in representation theory. Faddeev (with Reshetikhin and Takhtajan) gave a framework for quantum groups based on solutions of the Yang-Baxter equation, which followed closely the original ideas rooted in the theory of quantum integrable systems [47]. Nowadays, this approach is used in theory and applications of quantum groups and similar algebraic structures.

Another important idea created in the Lab at this time was the theory of form-factors in quantum integrable field theories. This lead to the implementation of Wightman's program of constructing relativistic quantum field theory axiomatically. The details can be found in [48]. We will not go into details of the work that was done in the Lab by Faddeev's students and colleagues. All this work was done under enormous intellectual influence of Ludwig with his encouragement and interest.

Other works

We did not give a complete account of Faddev's research. This is done in [49], so here we mention just a few of his other works.

Faddeev wrote important papers on the Riemann zeta-function and scattering, some with B. Pavlov. This was a very elegant attempt to give a spectral proof of the Riemann hypothesis. See [49] for a detailed description of this work.

Inspired by nonliear field theory and the interpretation of particles as quantization of localized solutions to Euler-Lagrange equations, he constructed nonlinear field theories with stable soliton-like solutions.

In 1986 Faddeev (with Shatashvili) returned to gauge symmetries and studied anomalies, with the key observation that anomalies are 1-cocycles for certain cohomology theories. (See [49] for details.) His desire to quantize gravity led to the study of the quantum Liouville equation, which was thought to be related to 2D gravity.

He had an important series of works on the quantum dilogarithm, the most important one with R. Kashaev [50]. This work led Faddeev to the construction of a modular double, the "double" of two Morita equivalent algebras [51]. Interestingly, the quantum dilogarithm first appeared in Faddeev's work on semiclassical quantization of solitons.

Recollections and opinions

Mathematical physics is a broadly defined field that emerged in 1950's-1960's. There are many views on what mathematical physics is. The consensus seems to be that it lies between mathematics and theoretical physics, having a non-empty overlap with both. At the same time, opinions vary on how much rigor should be present in mathematical physics research and how close it should be to physics.

According to Faddeev, the goal of a mathematical physicist is *not to make rigorous what is already understood, to the extent of being true beyond reasonable doubt, by physicists, but to achieve something they could not do with physical intuition, and do it on the basis of mathematical knowledge and mathematical intuition*.

In one of the articles aimed at a general audience, Ludwig noted: "If someone asked me who among the twentieth-century physicists impressed me most, I would answer: P. A. M. Dirac, H. Weyl, and V. A. Fock."

On the physics side he thought of himself as continuing in the tradition of V. A. Fock. He was in touch with the theory group at the Ioffe Physical Technical Research Institute, in particular with V. Gribov. In late 1970's and 1980's there were essentially two places in Leningrad where quantum field theory was developing: Faddeev's seminar, with more mathematical flavor, and Gribov's seminar, which was a physics seminar.

On the more mathematical side, as a student, he was under the strong influence of O. A. Ladyzhenskaya and was influenced by V. I. Smirnov's famous mathematical physics seminar in Leningrad. In many ways, Faddeev was a descendent of the St. Petersburg mathematical school, which goes back to Euler and includes such names as Ostrogradsky, Bunyakovsky, Chebyshev, Lyapunov, A. Markov, Krylov, Steklov, Smirnov, Vinogradov, Linnik, among many others.

Being one of the first mathematical physicists in the modern sense of this notion was very rewarding, but at the same time it was not easy. He was between two camps: physics and mathematics. Relations between mathematicians and physicists in the Soviet Union were sometimes complicated. Physicists had a lot of political leverage during the first, nuclear, period of the Cold War. The importance of applied mathematics and numerical methods was recognized slightly later with the development of rockets and the exploration of outer space. It is a well-

known anecdote that in the late 50's P. Kapitza and L. Landau were joking: "Where should we move the Division of Mathematics from the Academy of Sciences? Perhaps to the Committee for Sport, somewhere closer to chess?" That was the attitude of the time towards mathematics and mathematical directions in theoretical physics.

Ludwig mentioned in [52] and in the interview [53] that in 1960's, when mathematical physics, as a modern notion, was just emerging, he was very pleased to be in the company of his friends F. Berezin, V. Maslov, and R. Minlos, who were like-minded mathematicians with a deep interest in physics.

A. Sedrikyan, V. Garzadyan, A. Polyakov, A. B. Migdal, R. E. Kallosh, A. A. Migdal and L. D. Faddeev at a conference in Tsaghkadzor, Armenia, 1983.

Ludwig had many friends and colleagues around the world. P. Lax, L. Nirenberg, I. Singer and J. Moser were among mathematicians whose friendship Ludwig particularly valued.

To the question about the directions in mathematical and theoretical physics Faddeev always answered: "there is only one direction and one goal: understanding of the structure of matter and space-time."

L. D. Faddeev and I. Singer, Cambridge, Massachussets.

Work for science in the Soviet and post-Soviet period

All his life Ludwig worked at LOMI. He travelled a lot, especially after the end of the Cold War, but he was more or less free to travel even before, which was great for the Lab: we were getting fresh news. Of course, at the time it was very unfortunate that many other outstanding scientists in the Soviet Union were not allowed to travel abroad.

In 1972, he became the head of the Laboratory of Mathematical Methods in Theoretical Physics in the Leningrad Branch of the Mathematics Institute of the Academy of Science (LOMI) in Moscow. In 1976 he was elected a full member (academician) of the Soviet Academy of Sciences, and became the director of LOMI.

There is an interesting episode which took place immediately after these events. He was called to the local Communist Party committee and was rather pointedly asked why he was such a prominent figure but not a Party member. He answered that he was so busy working on mathematics that he did not have time to prepare to join in the Party [15]. To an urgent invitation, he replied that he does not want to, because then he will be asked to do a lot of administration and this will slow down his work. With some resentment local party bosses said OK, but every year a report on the scientific progress had to be submitted to the local party headquarters, which is what the scientific secretary of LOMI A.P. Oskolkov did for many years.

To be a director of any academic institute in the Soviet Union after 1968 was not easy. It was particularly difficult to be a director of a branch of the Mathematical Institute with headquarters in Moscow, where all the final decisions would be made anyway. Ludwig did

[15] Privately, he was very proud that neither he nor anyone in his family were ever Party members.

his best to maintain the excellence of the Institute during these somewhat administratively challenging times.

From 1986 till 1990 Faddeev was the president of the International Mathematical Union, which was a great distinction and a manifestation of the high prestige of Soviet mathematics.

Mathematical Physics Group at the Physics Department of Leningrad University, 1984. Sitting from left to right Faddeev, Birman, Bouldyrev. Faddeev was the Head of the Group during 1973-2001.

During the catastrophic dismantling of the Soviet Union, State-funded science became the first victim of privatization. Around this period many scientists left the Soviet Union or left science, as it was impossible to make a living on the meager salary offered for researchers. Ludwig was one of the very few people of his stature who did not leave the country for long periods at a time and he continued to work. Ludwig was offered at the time the directorship of the Institute for Theoretical Physics at Stony Brook, where the director was C. N. Yang, who was about to retire. Although Ludwig was very pleased with the offer, he declined it.

During this period, he was active in organizing exchange programs which would support young scientists remaining in St. Petersburg. One such program, which is remembered by many, was the exchange program with the University of Helsinki. Ludwig was also given a special grant from the Finnish Academy of Sciences to stay and work in Helsinki at any time in the early 90's. According to him it gave him a nice refuge during several years when he most needed it. He recalled this offer with gratitude many times.

From 1988 Ludwig was working on the creation of the Euler International Mathematical Institute in St. Petersburg. Against all odds the Institute started to operate in 1990, its renovated building opening in 1992. In 1992, he was elected Academician-Secretary of the Division of Mathematics of the Russian Academy of Sciences (known as the Division of Mathematical

Sciences after 2002)- a position he kept until his final days. This position allowed him to support the St. Petersburg mathematical community during those difficult times.

Ludwig was a great teacher. He gave an outstanding lecture series on quantum mechanics for mathematicians which resulted in a wonderful textbook for mathematics undergraduates [21]. His Thursday Seminar at LOMI was the center of modern mathematical physics in Leningrad and St. Petersburg. He wrote many survey articles which became standard reading material. One example, as mentioned before, was his lectures given in Les Houches in 1976 [38].

Recognition

For his numerous contributions to science, Ludwig Faddeev was elected to leading academies including the Royal Academy of Sweden (1989), the National Academy of the USA (1990), the French Academy of Sciences (2000) and the Royal Society of London (2010). He was awarded many prizes and other distinctions, among which were the Dirac Medal (1995), the Max Planck Medal (1996), the Euler Medal (2002), the Henri Poincaré Prize (2006), the Shaw Prize (jointly with V. Arnold, 2008), and the Lomonosov Medal (2014).

Farewell

Ludwig will be missed by his friends, pupils, and colleagues, for whom he was a constant source of inspiration for his passion for mathematics and physics.

Ludwig Faddeev is survived by his wife Anna M. Veselova, his daughters Maria Faddeeva and Elena Evnevich, his grandchildren Elena and Sergej Voklovy and Maria and Nikolai Evnevich, and by their families.

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XIXth International Congress on Mathematical Physics
Montreal, July 23-28, 2018
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On its three year cycle, the International Congress on Mathematical Physics (ICMP) is the most important conference of the International Association of Mathematical Physics (IAMP). In 2018 the Congress will return to the North America for the first time since 1983: the ICMP 2018 will be held in Montreal, July 23-28, 2018, at the Conference Centre Mont-Royal. The organizational aspects of the Congress are handled by the Canadian Mathematical Society.

The organization of the Congress is well under way. The list of plenary speakers and thematic sessions can be found at the Congress web site https://icmp2018.org/. As customary, the Congress will be preceded by the Young Researcher Symposium that will be held at McGill University July 20-21, 2018.

The Congress registration and financial aid applications for junior participants are open and can be accessed via its web site. The Congress will be accompanied by six satellite meetings which will be held the week before or after the Congress at BIRS, CRM, Fields Institute, Perimeter Institute, and McGill. Finally, the ICMP 2018 will be followed by the Fall 2018 CRM Thematic Program Mathematical Challenges in Many-Body Physics and Quantum Information http://www.crm.math.ca/crm50/en/activities/2018-activities/.

Canada and Montreal are proud to host the world of mathematical physics in 2018. The success of the Congress will be a further testament to the strength and vitality of our community. On the behalf of the IAMP and the Local Organizing Committee, I would like to invite all the members of the IAMP to attend the ICMP 2018.

Vojkan Jaksic
Professor of Mathematics, McGill University
Convenor of the ICMP 2018

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. Mr. Felix Hänle, LMU Munich, Germany
2. Dr. Hans Konrad Knoerr, Aalborg Universitet, Denmark
3. Mr. Konstantin Merz, LMU Munich, Germany
4. Prof. Thomas Kriecherbauer, Universität Bayreuth, Germany
5. Dr. Jonas Lampart, Université de Bourgogne, France
6. Prof. Kenneth McLaughlin, Colorado State University, USA
7. Prof. Kuldip Singh, National University of Singapore, Singapore
8. Mr. Daniele Dimonte, SISSA Trieste, Italy
9. Dr. Mutlay Dogan, Ishik University Erbil, Iraq
10. Dr. Suzanne Lanéry, Universidad Nacional Autónoma de México, Morelia, Mexico
11. Dr. Lorenzo Zanelli, University of Padova, Italy
12. Mr. Per von Soosten, TU Munich, Germany
13. Ms. Emanuela Laura Giacomelli, University of Rome, Sapienza, Italy

Recent conference announcements

Texas Analysis and Mathematical Physics Symposium 2017

Nov. 3-5, 2017. The University of Texas at Austin, USA.

Organized by T. Chen, D. Damanik.

https://www.ma.utexas.edu/users/tc/TeXAMP/TeXAMP-2017/TeXAMP-2017.html

Mathematical Challenges in Quantum Mechanics

Feb. 19-24, 2018. University of Roma, La Sapienza, Italy.

Organized by M. Correggi, D. Finco, G. Panati, A. Teta.

http://mcqm18.cond-math.it/

The Arizona School of Analysis and Mathematical Physics

March 5-9, 2018. The University of Arizona, Tucson, AZ USA.

Organized by H. Abdul-Rahman, R. Sims, and A. Young.

http://math.arizona.edu/ mathphys/AZSchool18

IST Austria Summer School in Probability and Mathematical Physics

June 4-8, 2018. Institute of Science and Technology Austria.

Organized by L. Erdős, J. Maas, R. Seiringer.

http://ist.ac.at/pmp2018

Open positions

Postdoctoral Position in Theory of Topological Meta-Materials, Yeshiva University (New York)

This position is in Prof. Emil Prodan's research group, in the Physics Department at Yeshiva University in New York. The position opens immediately and ends in June 30th, 2019. The position requires no teaching.

The deadline for applications is Nov. 1. To initiate your application, please send your CV and list of publications to Prof. Emil Prodan at prodan@yu.edu. The applicants deemed to qualified will be contacted and asked for letters of recommendations.

Postdoctoral position in random matrix theory in Bielefeld (Germany)

Applications are invited for a postdoctoral position in the area of random matrix theory. The successful applicant will work on a joint research project of Professors Gernot Akemann and Friedrich Goetze, preferably on recent developments in beta-ensembles, non-Hermitian random matrices and products thereof, or related areas.

The appointment is for a fixed-term three-year contract and will normally start in spring 2018, with an earlier or later date being possible. Prerequisite is a PhD or equivalent in Mathematical Physics or Mathematics, preferably related to random matrices. The salary will be very competitive (TVL E13 - E14).

The deadline for applications is Nov. 15, 2017. For further information visit the iamp webpage below or contact G. Akemann [akemann@physik.uni-bielefeld.de], or Friedrich Goetze [goetze@math.uni-bielefeld.de].

For more information on these positions and for an updated list of academic job announcements in mathematical physics and related fields visit

`http://www.iamp.org/page.php?page=page_positions`

Benjamin Schlein (IAMP Secretary)

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