Henri Poincaré Prize of the IAMP
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Oded Schramm (Microsoft Research)

Citation: For his contributions to discrete conformal geometry, where he discovered new classes of circle patterns described by integrable systems and proved the ultimate results on convergence to the corresponding conformal mappings, and for the discovery of the Stochastic Loewner Process as a candidate for scaling limits in two dimensional statistical mechanics.

Laudatio
delivered by Michael Aizenman (Princeton Univ.) at the
International Congress of Mathematical Physics, Lisbon 2003.

It is a pleasure to introduce as the prize winer an author of many beautiful works, and the inventor of the SLE process, which has yielded an effective computational tool and provided a refreshingly new perspective on the stochastic geometry of critical phenomena in two dimensions.

Oded Schramm was born in Israel slightly more than fourty years ago (Dec. 1961). He received his undergraduate education at the Hebrew University in Jerusalem (B.Sc. 1986, M.Sc. 1987), and following that pursued graduate studies at Princeton University (Ph.D. 1990), advised there in research by W.P. Thurston. He held research and academic appointments at Univ. Cal. San Diego (92), the Weizmann Institute (92-99), and Microsoft Research (since 99).

Much of Oded's work has involved analytical structures, probabilistic models, and stochastic geometry - where the above two areas meet. His early contributions include the proof, with Z.-X. He, of the countable case of P. Köbe's conjecture concerning conformal uniformization of multiply connected domains by circle domains, and significant advances related to the representation of analytic functions through circle packings along the approach of W.P. Thurston to the Riemann mapping theorem. Subsequently, he has produced interesting results concerning percolation models and other random systems. However, he is at present particularly celebrated for the introduction of the *Stochasic Loewner Evolution*. This family of processes (which many others refer to as the *Schramm-Loewner Evolution*), was discovered while searching for a convenient expression of the conformal invariance which is expected of the scaling limits of various two dimensional stochastic geometric systems.

Through the SLE, Schramm's name is destined to be linked with that of Charles Loewner. The namesake of the process belongs to a generation of analysts who have left strong marks on various branches of mathematics, yet who were caught in and affected to the extreme by the convulsions of Europe in the twentieth century. (A brief biographical sketch can be found at the fascinating St. Andrews 'History of Mathematics' electronic archive [1].)

Loewner's relevant work, which was motivated by issues in the theory of analytic maps, concerned the reconstruction of a curve in the complex plane from a conformally encoded data. Let $\gamma : [0,\infty) \mapsto \mathbb{C}$ be a curve forming a slit of the open unit disk D, growing from a boundary point into the interior. For each $t \geq 0$, let $g_t : D\backslash\gamma[0,t] \mapsto D$, be the Riemann map of the partly slit disk onto D, which is made unique by imposing the conditions: i. $g_t(0) = 0$, and ii. $g_t'(0)$ is real and positive.

As the slit is "unzipped" the point of growth, $\xi(t) = g_t(\gamma(t))$, is moving along the boundary ∂D. C. Loewner presented a procedure for solving the inverse problem: of recovering the curve $\gamma(t)$ from $\xi(t)$ and the rate information given by $g_t'(0)$ (which through reparametrization can be set to e^t).

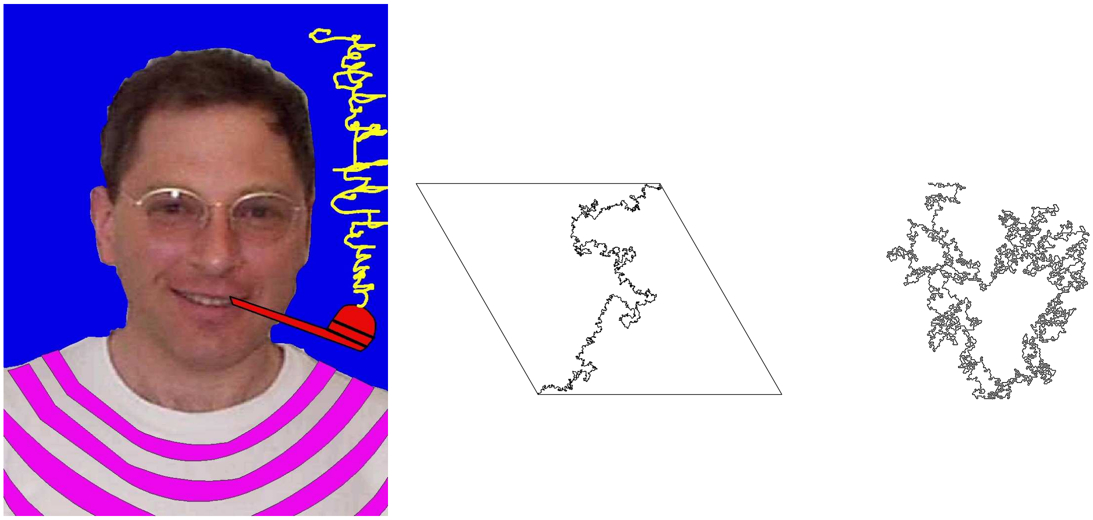
Oded Schramm has applied Loewner's prescription in considering random curves with the property that the conditional distribution of a future segment of γ, conditioned on the past, depends in a conformally invariant way on the past trajectory. He noticed that such a "conformal Markov property" requires $\xi(t)$ to evolve through independent increments, and thus to form a Brownian motion on the unit circle, at some fixed diffusion rate κ ($\left\langle [d\xi(t)]^2 \right\rangle = \kappa \, dt$). This has led him to formulate the one-parameter family of random paths SLE_κ, and show that under natural assumptions conformally invariant processes are described by this process, at suitable value of the parameter κ. The parameter is significant, and it affects various properties of the random process, including: the path's Hausdorff dimension, existence or not of self intersections, and the possibility of space filling ([3]).

Many beautiful results involving the SLE_κ were subsequently derived in collaborations with G. Lawler and W. Werner. These include a proof of an old conjecture of M. Mandelbrot about the dimension of Brownian motion's external perimeter, and a rigorous derivation of the more complete list of exponents discovered earlier in the work of B. Duplantier and Kwon [5], which determine the full mulitfractal spectrum of the Brownian motion. With the help of another insightful work of S. Smirnov [6], the exponents could be extended to percolation clusters as well.

It appears that the SLE_κ processes capture the scaling limits of a rich collection of two-dimensional models, including: boundaries of critical percolation clusters ($\kappa = 6$), Q-state Potts models and Fortuin-Kasteleyn random cluster models (κ varying with Q), self avoiding walks ($\kappa = 8/3$), frontier of brownian motion (conjectured by B. Mandelbrot to be related to the latter), loop erased random walks ($\kappa = 2$), paths mediating the spin-spin correlation in the ground state of the Heisenberg quantum anti-ferromagnetic spin chain [7] ($\kappa = 4$), and the uniformly sampled spanning trees (Peano paths, $\kappa = 8$).

The SLE processes provide a direct description of the continuum objects, e.g., the *scaling limit* of the self-avoiding random walk, bypassing the detailed analysis of the convergence – which is expected to take place but which in most cases is still beyond our analytical control. The basis for the SLE_κ construction are two features which facilitate analysis in two dimensions: the fact that *boundaries* of clusters are *paths* and the ubiquity of *conformal invariance*, at criticality. The latter has in the past found its expression in *conformal field theory*, and it is interesting to note that insights inspired by SLE are now being incorporated in new developments also in this area.

Let me end by noting that the work of Oded Schramm, and collaborators, is an excellent example of mathematical physics at work: mathematical results relate to insights and challenges coming from physics, but they are derived not by "crossing the t s" and "dotting the i s", but through the introduction of novel ideas which add a new perspective and enrich our understanding.



Oded Schramm, and some of his favorite processes (chordal SLE_4, and SLE_6).

References

[1] Biographical sketch of C. Loewner: St. Andrews electronic archive 'MacTutor History of Mathematics', http://turnbull.mcs.st-and.ac.uk/

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[3] S. Rohde and O. Schramm, Basic properties of SLE. to appear in *Annals of Math.*

[4] G. Lawler, O. Schramm, and W. Werner, Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.* **187**, 237 (2001). [Followed by parts II, III, and other works.]

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[7] M. Aizenman and B. Nachtergaele, Geometric aspects of quantum spin states. *Comm. Math. Phys.*, **164**, 17 (1994).

(The list is not intended to represent the wealth of past and recent works on the topics mentioned here. Figures - courtesy of O. Schramm.)