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Cover picture: Elliott Lieb, Krzysztof Gawędzki, and Antti Kupiainen

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Contents

Mathematical physics in the limelight

It has been a good season for recognition of mathematical physics, beginning with Giorgio Parisi's Nobel Prize in October. In this issue we are pleased to share recent announcements of the award of the Dannie Heineman Prize to Krzysztof Gawędzki and Antti Kupiainen, and of the APS Medal for Exceptional Achievement to Elliott Lieb, and to offer our congratulations.

Sadly, as we went to press we learned that Krzysztof Gawędzki passed away on 21 January. A public ceremony in his tribute will take place on Friday, 4 February, at 2:30 p.m. at the Lyon-Bron crematorium, 161 Bd de l'Université, 69500 Bron, France, and an online guestbook to collect testimonies and memories has been posted at the following address:

https://hommage.uneroseblanche.fr/bygiqb

We hope to honor him with a scientific appreciation in a future issue of the *News Bulletin*.

Elliott Lieb served twice as the President of the IAMP, and 25 years ago, during one of those terms, he moved the *News Bulletin* to electronic publication in March of that year. At that time he wrote to the membership:

> (3). Communication: The world is turning a corner; eventually, most scientific communication will be done electronically. This has not fully taken place for journals yet, but it is on its way. Many members have complained in the past that they did not always get the *News Bulletin* or ballots and this is partly due to the fact that ordinary 'snail' mail is not totally reliable, but more importantly that members do not keep their addresses up to date. In cooperation with Charles Radin, the new Secretary, I will try to make at least part of the *News Bulletin* available electronically, either on the web or by email.

> This will eventually be the standard practice; it is cheaper, quicker and more reliable than snail mail and it follows people around when they change their physical addresses. The details have to be worked out. Admittedly, some members might find email unreliable, and arrangements will be made to continue with hard copies in those cases. Email also allows the possibility for direct and rapid communication between the officers of the Association and members and I hope it will increase the sense of participation by members.

Elliott will be fêted next summer with a 90th birthday conference.

Evans Harrell, Chief Editor of the IAMP Bulletin

Krzysztof Gawędzki, Antti Kupiainen share 2022 Dannie Heineman Prize for Mathematical Physics

WASHINGTON, November 24, 2021 – The American Institute of Physics and the American Physical Society announce Krzysztof Gawędzki and Antti Kupiainen as the recipients of the 2022 Dannie Heineman Prize for Mathematical Physics. The prize is awarded annually by AIP and APS to recognize significant contributions to the field of mathematical physics.

”This award is honoring the work we did together in the 1980s and 1990s,” said Kupiainen, a Finnish professor of mathematics at the University of Helsinki. “This was a very fruitful and inspiring collaboration that meant a lot to me then and formed the basis also for my later career. Therefore, I am particularly happy of being awarded a prize for it.”

Gawędzki, a mathematical physicist at École Normale Supérieure in Lyon, and Kupiainen worked together to develop rigorous renormalization group methods for quantum field theory and statistical physics and made seminal contributions to conformal field theory and the Wess-Zumino-Witten-Novikov models. Later in the ’90s, they described and quantified anomalous scaling behavior in the turbulent advection of a scalar field.

”We are thrilled to announce the selection of Krzysztof Gawędzki and Antti Kupiainen as this year’s winners of the 2022 Dannie Heineman Prize for Mathematical Physics,” said Michael Moloney, CEO of AIP. ”Their work together on constructive quantum field theory and statistical mechanics energized the field, opening the door for further work into nonequilibrium statistical mechanics and turbulent flow problems in hydrodynamic models.”

The citation on their award reads: ”for fundamental contributions to quantum field theory, statistical mechanics, and fluid dynamics using geometric, probabilistic, and renormalization group ideas.” The prize will be presented at either the APS March Meeting in Chicago or the APS April Meeting in New York City.

Gawędzki was born in Żarki, Poland, received his doctorate degree from the University of Warsaw in 1971, and continued as a researcher at the Department of

Mathematical Methods in Physics in Warsaw. The martial law in Poland in 1981 forced him to emigrate, and he was invited first to the Institut des Hautes Études Scientifiques (IHES), where he stayed until 1999 and then to the École Normale Supérieure in Lyon, where he is currently a professor emeritus.

Kupiainen was born in Varkaus, Finland, and received his doctorate degree from Princeton University in 1979. After postdoctoral work at Harvard University, the Institute for Advanced Study in Princeton and IHES, he was appointed professor of mathematics in 1988 at Rutgers University and in 1991 at the University of Helsinki, where he is currently serving.

In the beginning of the '80s, Gawędzki and Kupiainen began their collaboration into the mathematical foundations of quantum field theory, which had been actively pursued by mathematical physicists. They arrived at a good understanding of so-called super-renormalizable theories that had been achieved by the pioneering work of James Glimm, Arthur Jaffe, Tom Spencer, Barry Simon, and many others. However, the theories of high-energy physics were not super-renormalizable, and the existing techniques were not sufficient to understand them.

The duo turned to the renormalization group method used in physics to study phase transitions, and they developed it into a tool for mathematically rigorous analysis of quantum field theory and statistical mechanics, in particular to construct certain renormalizable theories.

"Renormalization group is now a standard tool in mathematical physics, and I think our work had a role in making this happen," Kupiainen said.

In their later collaborations in the '90s, Gawędzki and Kupiainen noticed that certain questions related to turbulence were ripe for an analysis using techniques from quantum field theory. These problems dealt with advection of a passive scalar quantity like temperature or pollutant concentration by a turbulent flow.

They succeeded in showing by using quantum field theory methods that the statistics of the scalar show intermittent behavior and deviate from a Kolmogorov-type of scaling theory. This was the first such result in a turbulent system and influenced the way people thought about intermittency in standard fluid turbulence.

PROFILES OF THE WINNERS

Krzysztof Gawędzki, Ecole Normale Supérieure
Winner of the 2022 Dannie Heineman Prize for Mathematical Physics

Citation: *"For fundamental contributions to quantum field theory, statistical mechanics, and fluid dynamics using geometric, probabilistic, and renormalization group ideas."*

Background: Krzysztof Gawędzki (born in 1947 in Żarki, Poland) received his Ph.D. from the University of Warsaw in 1971 under Krzysztof Maurin, and then continued as a researcher at the Department of Mathematical Methods in Physics in Warsaw. In 1979-1980 he was invited to Harvard as an instructor, and there he worked in Arthur Jaffe's group. The martial law in Poland in 1981 forced him to emigrate and he was invited to IHES where, after obtaining a position at CNRS in 1982, he stayed until 1999. In 2000 he was appointed a professor at the École Normale Supérieure in Lyon, where he is currently a professor emeritus. In 1986 he was an invited speaker at the International Congress of Mathematics in Berkeley. His main field at the beginning of his career was quantum field theory. Together with A. Kupiainen, he developed rigorous renormalization group methods for quantum field theory and statistical physics and made seminal contributions to conformal field theory, in particular to the Wess-Zumino-Witten-Novikov models. Later in the 90's he concentrated on the studies of turbulence; together with Kupiainen he described and quantified anomalous scaling behavior in the advection of a scalar field. In that field he also had fruitful collaboration with G. Falcovich and M. Vergassola. In recent years, Gawędzki made seminal contributions to non-equilibrium statistical mechanics, in particular to fluctuation theorems and the theory of entropy production, as well as to the theory of topological, in particular Floquet, insulators.

Antti Kupiainen, University of Helsinki
Winner of the 2022 Dannie Heineman Prize for Mathematical Physics

Citation: *"For fundamental contributions to quantum field theory, statistical mechanics, and fluid dynamics using geometric, probabilistic, and renormalization group ideas."*

Background: Antti Kupiainen is Professor of Mathematics at the University of Helsinki. He received his M.S. in mathematics from Helsinki University of Technology in 1976 and his Ph.D. in physics from Princeton in 1979 under the supervision of Tom Spencer and Barry Simon. After postdoc work at Harvard in Arthur Jaffe's group and periods at the Institute for Advanced Study at Princeton, Harvard and IHES in Paris he became Professor of Mathematics at Rutgers University where he benefitted from the inspired atmosphere of Joel Lebowitz's group. Since 1992 he has been employed by the University of Helsinki and the Academy of Finland. He is a member of the International Association of Mathematical Physics (IAMP) where he served as the president in 2012-14. He has been twice the recipient of the European Research Council (ERC) advanced grant and twice an invited speaker at the International Congress of Mathematics (ICM). Professor Kupiainen has worked on mathematical problems connected to quantum field theory, statistical mechanics, turbulence and most recently on Liouville quantum gravity. He has enjoyed and benefited from working in collaboration with several excellent scientists: most prominently Krzysztof Gawędzki and Jean Bricmont and most recently Remi Rhodes and Vincent Vargas. His major accomplishments include rigorous construction of asymptotically free quantum fields, proof of phase transition in random field Ising model, discovery of intermittent scaling behaviour in turbulent advection and proof of the DOZZ formula for Liouville conformal field theory.

The Selection Committee: Jan Dereziński (chair), Joel Lebowitz, Herbert Spohn, Hal Tasaki, and Nicolai Reshetikin

ABOUT THE HEINEMAN PRIZE

The Heineman Prize is named after Dannie N. Heineman, an engineer, business executive, and philanthropic sponsor of the sciences. The prize was established in 1959 by the Heineman Foundation for Research, Education, Charitable and Scientific Purposes, Inc. The prize will be presented by AIP and APS on behalf of the Heineman Foundation at a future APS meeting. A special ceremonial session will be held during the meeting, when Gawędzki and Kupiainen will receive the $10,000 prize.
http://www.aps.org/programs/honors/prizes/heineman.cfm

ABOUT AMERICAN INSTITUTE OF PHYSICS

The American Institute of Physics (AIP) is a 501(c)(3) membership corporation of scientific societies. AIP pursues its mission – to advance, promote, and serve the physical sciences for the benefit of humanity – with a unifying voice of strength from diversity. In its role as a federation, AIP advances the success of its Member Societies by providing the means to pool, coordinate, and leverage their diverse expertise and contributions in pursuit of a shared goal of advancing the physical sciences in the research enterprise, in the economy, in education, and in society. In its role as an institute, AIP operates as a center of excellence using policy analysis, social science, and historical research to promote future progress in the physical sciences.

ABOUT AMERICAN PHYSICAL SOCIETY

The American Physical Society is a nonprofit membership organization working to advance and diffuse the knowledge of physics through its outstanding research journals, scientific meetings, and education, outreach, advocacy, and international activities. APS represents more than 50,000 members, including physicists in academia, national laboratories, and industry in the United States and throughout the world. https://www.aps.org/

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2022 APS Medal for Exceptional Achievement in Research awarded to Elliott Lieb

by David Barnstone

Mathematical physicist Elliott H. Lieb has been selected to receive the 2022 APS Medal for Exceptional Achievement in Research for "major contributions to theoretical physics through obtaining exact solutions to important physical problems, which have impacted condensed matter physics, quantum information, statistical mechanics, and atomic physics."

Elliott Lieb

Awarded annually, the Medal is the highest honor the Society bestows upon researchers across all of physics, recognizing contributions of the highest level that advance our knowledge and understanding of the physical universe in all its facets. The recipient will be recognized for seminal contributions to several fields of physics at a ceremony during the APS Annual Leadership Meeting on January 27.

Lieb is lauded by colleagues and peers for his rigorous mathematical approach to solving fundamental problems in physics. Among his hundreds of scientific publications is one of the most-cited papers in condensed matter physics on the one-dimensional Hubbard model, published in Physical Review Letters in 1968.

Lieb is also known for his solution to the "square ice problem," or the number of possible configurations of hydrogen atoms in a lattice of water molecules. This solution started a significant subfield in statistical mechanics. Some other major contributions include the strong subadditivity of quantum entropy, the Thomas-Fermi theory of atoms, the Lieb-Robinson velocity, the AKLT Spin Model, and the Lieb lattice for ferrimagnetism.

"With this prize we recognize Elliott's lifetime of accomplishments that have transformed physics," said APS President Frances Hellman, who chaired the 2022 Selection Committee. "It is a celebration of his dedication to scientific inquiry and pursuit of knowledge."

Lieb obtained his bachelor's degree in physics from the Massachusetts Institute of Technology in 1953 and his PhD in mathematical physics from The University of Birmingham in 1956. He has held research and faculty positions at the University of Illinois, Cornell University, IBM, Yeshiva University, and MIT. He has been a professor at Princeton University since 1975.

"I will be delighted to honor Elliott with the APS Medal at our Annual Leadership Meeting in Washington, DC in January," said APS CEO Jonathan Bagger. "His life and career have taken him across physics and around the world, yielding important discoveries at nearly every turn."

"Physics is a big enterprise with many people doing various things, being held together by a common interest in science," said Lieb. "It's important to have scientific institutes like the American Physical Society that bring all this together."

The Medal includes a $50,000 prize, a certificate citing the contribution made by the recipient, and an invited talk at an APS March or April Meeting. The prize is funded by a donation from entrepreneur Jay Jones.

The author is APS Head of Public Relations.

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Photo credit: E. Lieb.

Universality and integrability in random matrix theory and interacting particle systems

by Ivan Corwin, Percy Deift, Alice Guionnet, Alexander Its, Herbert Spohn

In 1999, 2010 and again in 2021 the Mathematical Sciences Research Institute in Berkeley, California, hosted highly successful programs focused on the fruitful interface between random matrix theory and other fields, in particular interacting particle systems. This article, written by some of the organizers of this semester, provides a brief and incomplete survey of some of the tremendous progress over the past decade in these areas, for instance in the robust theory of universality for random matrix eigenvalue statistics, or the rich integrable structure uncovered behind a host of interacting particle systems and random growth models. This text is closely adapted from an article published in the fall 2021 issue of Emissary (MSRI's newsletter) with the permission of MSRI and was supported by the National Science Foundation under Grant No. 1440140, the National Security Agency under Grant No. H98230-21-1-0060, and the Simons Foundation, while the authors were in residence at MSRI during fall 2021.

Random matrix theory has many roots, perhaps explaining why it has so successfully thrived as a research area bridging mathematics and many other disciplines (such as statistics, physics, computer science, data science, numerical analysis, biology, ecology, engineering, operations research). In statistics, Wishart began the study of sample covariance matrices in the 1920s. Quite separately in nuclear physics, Wigner introduced and studied certain Gaussian matrix ensembles invariant under classical symmetry groups (i.e. conjugation by orthogonal, unitary or symplectic matrices) in the 1950s (see Figure 1). Goldstein and von Neumann came upon random matrix theory at a similar time from the perspective of numerical analysis and estimation of condition numbers. In number theory, in a surprising development in the 1970s, Montgomery recognized that random matrix statistics described the non-trivial zeros of the Riemann Zeta function (see

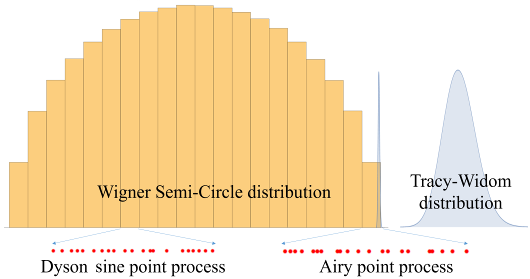

Figure 1: The Gaussian Unitary Ensemble (GUE) is a measure on $N \times N$ Hermitian matrices in which all diagonal entries are real Gaussian distributed and all off-diagonal entries are complex Gaussian distributed (all Gaussians being independent). The ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_N$ are consequently real and random, and their joint density (against Lebesgue measure on $\mathbb{R}^N$) is proportional to $\prod_{k=1}^{N} e^{-\lambda_k^2/2} \prod_{i<j}(\lambda_i - \lambda_j)^2$. For large N, the histogram of eigenvalues becomes well approximated by the *Wigner semi-circle* distribution. Zooming in to the resolution of individual eigenvalue, one sees the *Dyson sine point process* inside the support of the histogram and the *Airy point process* near the edge. In particular, the distribution of the maximal eigenvalue follows a *Tracy-Widom distribution.*

Figure 2). More recently, there have been a host of new motivations and sources for problems in random matrix theory, or new uses of the tools which have been developed in its study. It is this constant growth and expansion of the field which has made it one of the most dynamic and exciting areas of mathematics.

While some applications of random matrix theory techniques come quite naturally, others (like the number theory ones mentioned above) come as a surprise and take a while to fully develop. In the late 1990s, such a mysterious link was discovered between random matrix ensembles and a few interacting particle systems, namely the longest increasing subsequence problem for random permutations and the closely related totally asymmetric simple exclusion process (see Figure 3). This linked random matrix theory to a vibrant and growing area of

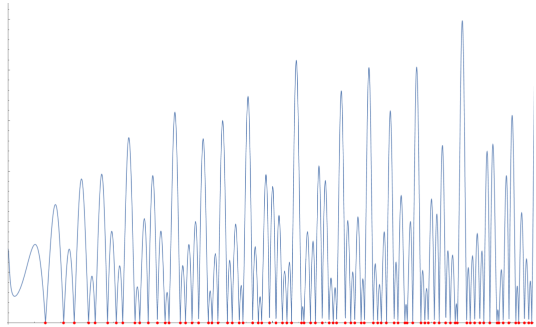

Figure 2: The plotted blue curve is $|\zeta(\frac{1}{2}+it)|$, where $\zeta(z)$ is the Riemann zeta function. The zeros form a point process, denoted in red, which is widely believed to share many properties with the zeros of random Hermitian matrices. For instance, recently there has been great interest in comparing the modulus of the zeta function on the critical line to the characteristic function for certain types of random matrices, and relating both to log-correlated Gaussian fields known as *Gaussian multiplicative chaos*. Based on this comparison, Fyodorov, Hiary and Keating conjectured a precise asymptotic for the maximum modulus of the zeta function on almost all intervals along the critical axis. Major progress in this direction has come in recent work of Arguin, Bourgade and Radziwill.

probability and non-equilibrium statistical mechanics, and led to a bevy of new problems, methods and results. The origins of the link have been progressively exposed over time and have further connected these fields to asymptotic representation theory, quantum integrable systems and algebraic combinatorics.

Interacting particle systems arise as probabilistic models of real-world systems such as traffic, queues, and mass transport; and through certain transforms or limits they also relate to random interface growth, random walks in random media, stochastic optimization problems, and stochastic PDEs. These types of systems have been actively studied since the 70s in probability, as well as other more applied fields including non-equilibrium statistical mechanics.

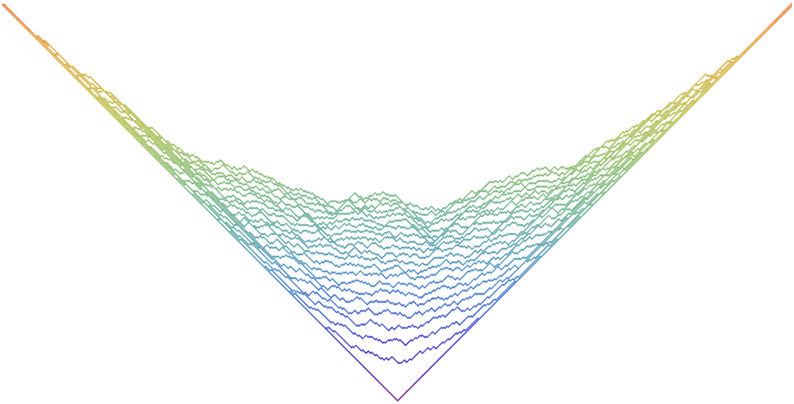

Figure 3: The temporal evolution of the height function for the totally asymmetric simple exclusion process is shown. Initially, the height function looks like $|x|$. Every local minimum can grow into a local maximum after independent exponentially distributed waiting times. Remarkably, the distribution of this process can be exactly linked to the distribution of the largest eigenvalue from the Laguerre Unitary Ensemble (LUE), a measure on complex sample covariance matrices. Figure by Leonid Petrov.

Within random matrix theory, and more broadly probability and statistical mechanics, there are often two complementary themes – universality and integrability. Universality refers to the idea that randomness smooths out microscopic differences between systems and hence only certain key phenomenological properties of a system will control the large-scale or long-time behavior. The simplest instance of this concept at play is the central-limit theorem for independent identically distributed random variables where, after fixing the mean and variance, all sums have the same universal Gaussian limit. Integrability (or sometimes exact solvability) refers to the search for models which enjoy enhanced algebraic structure which enables exact calculations and precise asymptotics. Indeed, with the central limit theorem example, coin flipping admits exact formulas in terms of binomial distributions which yielded for the first time (in 1738) the Gaussian distribution (long before it was proved universal around 1900). In a sense, universality says that many systems share a common limit, and integrability identifies precisely what that limit is.

In the context of random matrices, integrable models, such as the Gaussian Unitary Ensemble (mentioned in the caption to Figure 1), have statistics that can be written exactly in terms of compact formulas which are amenable to asymptotic analysis (generally, as the matrix size grows to infinity). Quite remarkably (as shown in works of Soshnikov and the works of Erdős, Yau, and co-authors, as well as Tao and Vu), these limiting statistics are quite robust – changing the underly distribution of matrix entries from Gaussian to other distributions does not change these limiting statistics.

Besides changing the underlying distribution of matrix entries, there are many other natural classes of random matrices to consider. For instance, in spectral graph theory one seeks to understand the eigenvalues of adjacency matrices for large classes of random graphs (see Figure 4). A challenging conjecture which now seems within reach is to prove that for a randomly chosen d-regular graph on N vertices, the eigenvalues in the bulk of the spectrum (i.e., away from the edges of the support) are universal and independent of d, down to the minimal non-trivial value of $d = 3$. For the d-regular random graph model as well as the Erdős-Reyni random graph model, there remain many such compelling questions, despite great progress over the past few years. Another form of universality deals with studying matrix ensembles which are still invariant under the action of classical symmetry groups, but for which individual entries are not Gaussian. This form of universality was established in work in the early 2000s of many authors, including Bleher, Deift, Gioev, Its, Kriecherbauer, Lubinsky, McLaughlin, Pastur, Scherbina, Venakides, and Zhou. Yet another direction is to move away from invariant ensembles to the realm of β-ensembles and log-gases, in which case universality has come from various works, including those of Bekerman, Borodin, Bourgade, Dumitriu, Edelman, Erdős, Figalli, Gorin, Guionnet, Leble, Ramirez, Rider, Serfaty, Sutton, Virag, Valko, and Yau.

Besides the intrinsic value in their study, random matrices also pop up in many remarkable applications including in the study of random non-linear func-

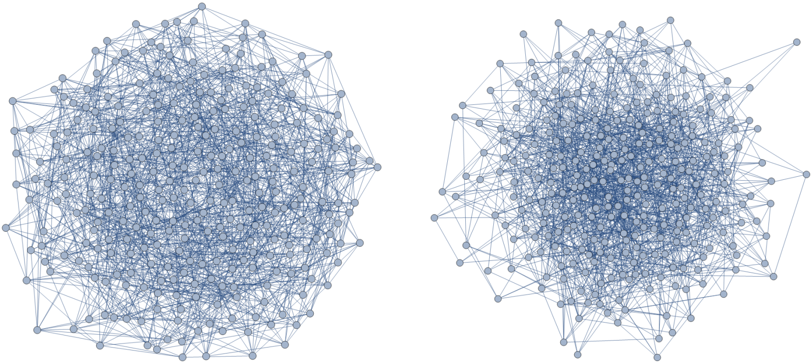

Figure 4: Two random graphs on $N = 300$ vertices. Both graphs have average degree 10, though the one on the left is uniformly chosen from 10-regular graphs while the one on the right is chosen according to the Erdős-Reyni measure on graphs.

tions of many variables. Examples include Hamiltonians of disordered models like spin-glasses (in statistical mechanics), or landscapes of inference problems in high-dimensional statistical estimation (in data science). A fundamental problem in both contexts is to understand the *complexity* of the topology of the landscapes of such random non-linear function of many variables (see Figure 5 for such a complex landscape in 2-dimensions). This landscape complexity is closely linked to the efficiency of natural exploration or optimization algorithms in these landscapes. If there are many valleys of differing heights, separated by large saddle-points, it will take these algorithms a long time to find the global minimum (or near-minimum). In high dimensions, the landscape complexity is controlled by the behavior of the Hessian, which itself is a random matrix. Thus, the toolbox of results and methods from random matrix theory become vital.

The tools and statistics which arise in the study of random matrices also play an important role in the study of many other probabilistic systems. One prime example is found in the realm of random tilings. In Figure 6 one starts with a hexagonal domain with a diamond cut out from the inside and then fills it with three types of rhombuses: ♦ and its $2\pi/3$ and $4\pi/3$ rotation. There are many (in

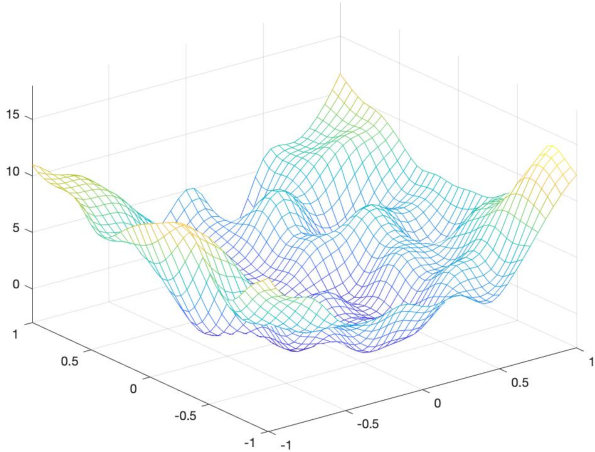

Figure 5: Though the main interest is in high-dimensional complex landscapes, here we see a 2-dimensional complex landscape where there are many valleys, saddle points and hills. Figure by Benjamin McKenna.

fact exponential in the system size) admissible tilings and one is chosen uniformly at random among these. Remarkably, for large system sizes the behavior of the tiling is quite stable to first order (i.e., there is a *limit shape*), and then shows smaller scale fluctuations. Recent work of Aggarwal has proved that the local statistics describing the tiling inside the limit shape are universal (with respect to changing the boundary of the domain) and related to discrete versions of point processes (such as the Dyson sine process) arising in random matrices. Other properties, such as Gaussian free-field global fluctuations of the height function around the limit shape have been shown in some special domains by Bufetov, Gorin, Kenyon and Petrov, though this type of result for general domains remains open. At the edge of the disordered region, Aggarwal and Huang have shown that the Airy process arises quite generally as well.

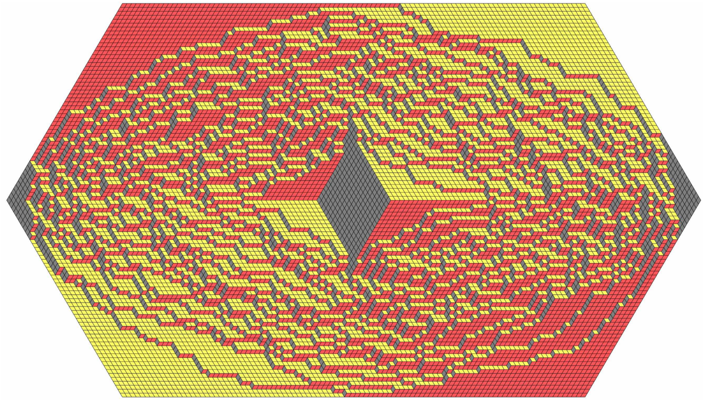

Figure 6: A random tiling of a hexagonal region with a deleted interior diamond. Figure by Leonid Petrov.

A decade ago, the study of the *Kardar-Parisi-Zhang* stochastic partial differential equation (or KPZ equation for short) experienced two major developments. Work of Amir, Calabrese, Corwin, Le Doussal, Rosso, Sasamoto, Spohn, Tracy, and Widom revealed that statistics could be computed exactly for this paradigmatic model for non-linear stochastic interface growth; and the work of Hairer, Gubinelli, Perkowski, Jara, Gonçalves, and others developed a robust solution theory for this equation. In the decade since then, the KPZ equation has attracted immense interest from both perspectives. The exact solvability of the KPZ equation and related models has been put into a more general context in the development of the field of *integrable probability*, featuring work of Borodin, Corwin, and collaborators on Macdonald processes and stochastic vertex models. Another set of notable developments in this area is the construction of the *KPZ fixed point* in work of Matetski, Remenik, and Quastel, and of the *directed landscape* in work of Dauvergne, Ortmann, and Virag. On the solution theory side, the methods of regularity structures and paracontrolled distributions and energy solutions have proved to be powerful and general tools, able to address other singular stochastic PDEs as well as to demonstrate the various regularization schemes all define the same solution theory.

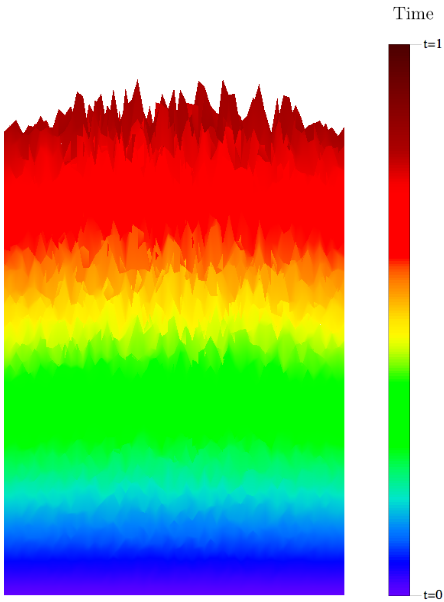

Figure 7: The evolution of a stochastic interface at different times (indicated through different colors). Figure by Alexandre Krajenbrink.

At the heart of integrable probability theory and the exact solution to the KPZ equation lie tools of *quantum integrable systems* such as the *Yang-Baxter equation*, *Bethe ansatz* and associated families of symmetric functions (e.g. Macdonald, Hall-Littlewood, Whittaker, Schur). These tools have been used in the study of quantum spin chains and two-dimensional equilibrium statistical mechanics for many decades, with origins in works from the 1930-1970s of Baxter, Bethe, Lieb, Sutherland, and Yang, and subsequent developments in the 19080s of Faddeev and collaborators, as well as Drinfeld and Jimbo in the form of quantum group theory. Figure 8 illustrates the *colored stochastic vertex model* introduced and studied recently by Borodin and Wheeler, relying on these tools.

Classical integrable systems have also long been known to relate to the asymptotic statistics that arise in the study of stochastic interface growth models, interacting particle systems, and random matrices. However, over the past few years there are some quite surprising links that have emerged.

The modern theory of classical integrable systems has its origin in the remarkable work of Gardner, Green, Kruskal, and Miura in 1967 and has since expanded to connect to the analysis of exactly solvable quantum field and statistical physics models, the theory of integrable nonlinear PDEs and ODEs, and quantum and classical theories of dynamical systems integrable in the sense of Liouville. Over the years integrable system theory has emerged as one of the principal sources of new analytical and algebraic ideas for many branches of contemporary mathematics and theoretical physics.

A key ingredient of the analytic part of the theory of integrable systems is the *Riemann-Hilbert Method* which reduces the problem at hand to a certain *Riemann-Hilbert problem* of analytical factorization of a *jump matrix* defined on some appropriate contour in the complex plane. As a result, the solution of the original, usually nonlinear, problem is given in terms of the solution of the abovementioned Riemann-Hilbert problem. A generic matrix Riemann-Hilbert problem cannot be solved explicitly in terms of contour integrals. It can, however, always be reduced to the analysis of a linear singular-integral equation. One may think of a Riemann-Hilbert representation of the solution of the original problem as a *non-abelian* analog of the integral representations of classical special functions, such as Bessel functions or Legendre functions. The great benefit of reducing an originally nonlinear problem to the analytic factorization of a given matrix function comes to the fore in the asymptotic analysis of the problem at hand, when one can exploit the *nonlinear steepest-descent* method which was introduced in 1992 by Deift and Zhou as a culmination of some 20 years of significant efforts of several authors in the development of an efficient scheme for the asymptotic analysis of oscillatory Riemann-Hilbert problems.

In the past two decades, several new areas have fallen under the realm of integrable systems. Among these areas an important place is occupied by random matrix theory. Indeed, the use of integrable techniques, notably the Riemann-

Hilbert method, has made it possible to solve some of the long-standing problems of the theory, such as universality for invariant matrix ensembles. In a reciprocal fashion, the involvement of integrable theory into random matrix theory has extended the methods of integrable systems even further to areas which previously have never been considered as integrable systems, and which include string theory, enumerative topology, random permutations, and number theory. In turn these new directions have posed new challenges to the analytical apparatus of integrable systems itself.

Of a particular interest is the extension of Riemann-Hilbert techniques to the study of the KPZ equation and the general-β Tracy-Widom distributions. The possibility to apply Riemann-Hilbert methods is based on the very recent discovery of the remarkable integrable structures of KdV and Painlevé-type behind these subjects. The Riemann-Hilbert setting which arises is quite unusual, and involves operator-valued and block-matrix Riemann-Hilbert problems. Another subject of great interest is the study of hydrodynamic limits for integrable systems with infinite conservation laws, such as the Toda chain, as well as correlation functions for quantum spin chains, all of which relate to both random matrix ensembles and statistics coming up in the study of the KPZ equation.

The topics highlighted above are just a sampling of the subjects of current intense study in the domains of random matrix theory and interacting particle systems. The balance of the perspectives and techniques of universality and integrability have proved to be a powerful dichotomy. Undoubtably, in a decade the landscape in this area will look quite different, though one constant will be the tremendous progress due to researchers across the field.

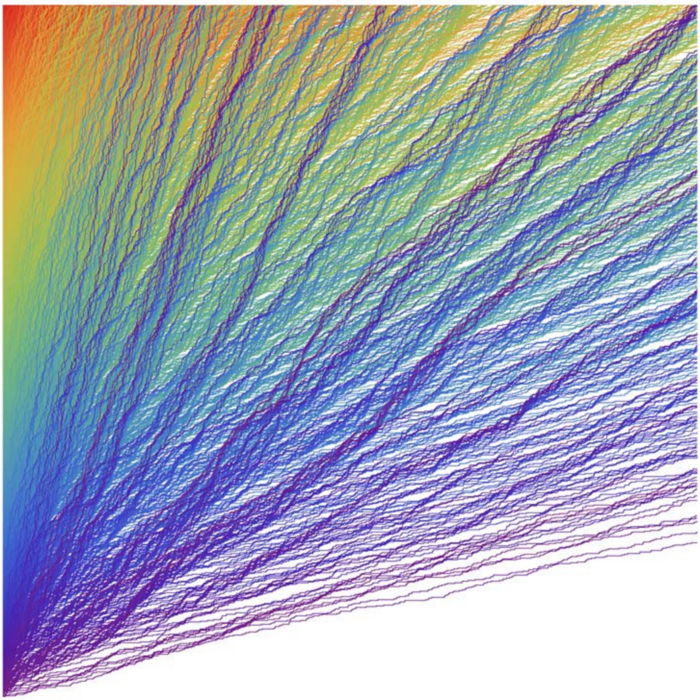

Figure 8: Colored paths emanate from the left boundary and move as random walks except when they intersect, in which case they switch orders with probabilities that depend on their relative ordering. The large-scale evolution of this system is displayed here. Figure by Leonid Petrov.

A personal experience in mathematical physics

by Demetrios Christodoulou

The purpose of the present note is to make known my personal experience in connection with the works for which I was awarded a Henri Poincaré Prize at the ICMP conference in Geneva. I shall discuss my motivation in studying these problems and how the ideas that led to their solution came to me. I hope that people may find something of interest in this discussion to which they may relate.

Chronologically the first was my work on the nonlinear memory effect in the theory of gravitational radiation [C1]. This was connected with, and was simultaneous to, my joint work with Sergiu Klainerman [C-K] on the global nonlinear stability of the Minkowski spacetime of special relativity in the framework of general relativity, where the spacetime is curved and the metric satisfies the Einstein equations. Both works are in the context of the vacuum Einstein equations which simply state that the Ricci curvature of the metric vanishes. Part of the motivation for studying the stability problem was to provide a rigorous basis for the theory of gravitational radiation. This required an analysis of the asymptotic behavior of the solutions, in particular the behavior at future null infinity. In fact, our approach to the stability problem itself required an understanding of the asymptotic behavior. This is because the approach was along the lines of the method of continuity and involved a bootstrap argument. We recall that a maximal spacelike hypersurface is a hypersurface with an induced Riemannian metric making it complete, such that any compact perturbation of the hypersurface results in a decrease in volume. We may assume that the initial data corresponds to such a maximal spacelike hypersurface. In the main part of the bootstrap argument one considers a spacetime slab, bounded in the past by the initial hypersurface, which is foliated by maximal spacelike hypersufaces and where certain bootstrap assumptions on a certain set of geometric quantities hold, and is the spacetime slab of the greatest temporal extent possessing these properties. Denoting by H_t the leaves of the foliation into maximal hypersurfaces, with t normalized at spatial infinity, either the spacetime slab is of infinite temporal extent and we have a solution complete toward the future and satisfying the bootstrap assumptions throughout, or else it is of finite extent t_*, and there is a last slice t_*. The main step in the argument is then to show that, provided the initial data satisfies a certain smallness condition, none of the inequalities involved in the bootstrap assumptions is saturated at H_{t_*}. Once this is established, the spacetime slab is shown to be extendable to a greater slab which also satisfies the bootstrap assumptions, thus contradicting the maximality of the original slab. This rules out the second alternative, leaving only the alternative of global existence toward the future.

Now, allowing the temporal extent of the spacetime slab in which the solution is to be under control to have no upper bound is possible if the solution possesses appropriate decay properties. To state these decay properties requires the introduction of an additional function besides the maximal time function t discussed above. This additional function is the optical function u and its level sets are outgoing null hypersufaces C_u. The decay is a reflection of the expansion of the C_u, that is the growth of the area $A(t,u)$ of the cross sections $S_{t,u} = H_t \cap C_u$ of each C_u by the H_t. With the aid of the two functions t and u the action the spacetime slab of an appropriate subgroup of the conformal group of Minkowski spacetime is constructed. This action is to approach that of conformal isometries at late times. For this reason the construction starts at the last slice H_{t_*}. Once a surface $S_{t_*,u}$, a level set of u on the last slice H_{t_*} is defined, the corresponding outgoing null hypersuface C_u is simply the inner component of the boundary of the past of $S_{t_*,u}$ in the spacetime slab. Using the vector fields generating the group action, certain 3-forms are constructed the integrals of which on the H_t and the C_u control the spacetime curvature. These are quantities whose boundedness encodes the decay properties and whose growth is controlled in terms of the same set of quantities. This is what allows the main step in the bootstrap argument to be completed.

It should be evident from the above discussion that the proper construction of the function u on the last slice H_{t_*} plays a crucial role. It was the study of this problem that eventually led me to the nonlinear memory effect. It was clear that the surfaces $S_{t_*,u}$ had to be chosen so that they become equidistant after the limit $t_* \to \infty$ is allowed to be taken. However, for given finite t_*, to define u on H_{t_*} to be the signed distance function from a given surface $S_{t_*,0}$ chosen to be the 0-level set of u on H_{t_*} is inappropriate because there is a loss of one degree of differentiability associated with the distance function. This problem was overcome by subjecting the surface $S_{t_*,u}$ to a suitable equation of motion of surfaces on H_{t_*} which determines them given the initial surface $S_{t_*,0}$ making them as smooth as $S_{t_*,0}$, and equidistant in the limit $t_* \to \infty$.

The definition of the function u in the spacetime slab reduces according to the preceding to the definition of the surface $S_{t_*,0}$ on H_{t_*}. In fact, $S_{t_*,0}$ can be defined starting from the initial hypersurface H_0 by first defining a surface $S_{0,0}$ on H_0 and then defining the outgoing null hypersurface C_0 to be the outer component of the boundary of the future of $S_{0,0}$ in the spacetime slab. Here it is assumed that the generators of C_0 so defined have no future end points in the interior of the spacetime slab, something which is established in the course of the continuity argument. The definition of u in the spacetime slab thus reduces to the choice of surface $S_{0,0}$, the 0-level set of u on H_0. Given that the initial data on H_0 are asymptotically flat, $S_{0,0}$ is to be chosen in accordance with the condition that the spacetime curvature along H_0 falls-off as we move far away from the interior of $S_{0,0}$ on H_0, but is otherwise essentially arbitrary.

Physical understanding is achieved only after the limit $t_* \to \infty$ is taken. The spacetime region between C_{-b} and C_b for some large but finite positive b can be thought of as the "wave zone", the region where the gravitational waves travel, the spacetime region exterior to C_{-b} representing the region before the passage of the waves and that interior to C_b the region after the passage of the waves. Now, a normal vector field L to a null hypersurface, such as a C_u, is tangential to C_u and the 2nd fundamental form χ of a section of C_u, such as a $S_{t,u}$, relative to the intrinsic normal L measures the deformation of the section under translation along the integral curves of L which are the null geodesic generators of C_u, the trace of χ relative to the induced metric on the section measuring the change in area of the section and the trace-free part $\hat{\chi}$, a trace-free symmetric tensor field on the section, measuring the change in shape. A spacelike surface, such as a $S_{t,u}$, has also another null normal $\underline{L}$, which in the case of $S_{t,u}$ is transversal to C_u, $\underline{L}$ being conjugate to L, that is we have $g(\underline{L}, L) = -2$ where g is the spacetime metric. To $\underline{L}$ there corresponds another 2nd fundamental form $\underline{\chi}$, extrinsic to C_u in the case of $S_{t,u}$. The main quantities associated to a spacelike surface in spacetime, such as $S_{t,u}$, are completed by the torsion ζ, which is the connection in the normal bundle of the surface, a $SO(1,1)$ connection, the normal planes to the surface being timelike planes. Now it turns out that considering the $S_{t,u}$ sections of a given C_u and setting $r = \sqrt{A/4\pi}$, A being the area of $S_{t,u}$, as $t \to \infty$ $r^2\hat{\chi}$ tends to a limit Σ, $r\underline{\hat{\chi}}$ tends to a limit Ξ and $r^2\zeta$ tends to a limit Z. Also, while $r\mathrm{tr}\chi$ tends to 2, setting $h = r\mathrm{tr}\chi - 2$, rh tends to a limit H. Here with a generator of a C_u corresponding to a point on $\mathbb{S}^2$, the unit sphere of directions, Σ and Ξ are symmetric trace-free 2-covariant tensor fields, Z is a 1-form on $\mathbb{S}^2$, and H is a function on $\mathbb{S}^2$, depending on u. These are all quantities at future null infinity, which can be identified with the manifold $\mathbb{R} \times \mathbb{S}^2$, a point on $\mathbb{R}$ corresponding to a value of u.

The above quantities are related by certain equations which are limits at future null infinity of corresponding equations in spacetime. In particular, there is the equation

$$\frac{\partial \Sigma}{\partial u} = -\frac{1}{2}\Xi \tag{1}$$

which represents the limit of the equation expressing the variation of $\hat{\chi}$ along $\underline{L}$, that is transversally to the C_u. The limit of the corresponding variation equation for h is simply

$$\frac{\partial H}{\partial u} = 0 \tag{2}$$

The leading component of the spacetime curvature toward future null infinity is the restriction of the $R(\cdot, \underline{L}, \cdot, \underline{L})$ part of the curvature tensor at each point to the tangent space to the $S_{t,u}$ to which that point belongs. This is a symmetric trace-free 2-covariant tensor field on the $S_{t,u}$, which we denote by $\underline{\alpha}$. As $t \to \infty$, $r\underline{\alpha}$ tends to a limit $\underline{A}$, a symmetric trace-free 2-covariant tensor field on $\mathbb{S}^2$ depending on u, and we have the equation

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{2}\underline{A} \tag{3}$$

There is also the following elliptic equation on $\mathbb{S}^2$ relating Σ to H and Z:

$$\overset{\circ}{\not{d}\mathrm{iv}}\; \Sigma = \frac{1}{2}\overset{\circ}{\not{d}}\, H + Z \tag{4}$$

Here $\overset{\circ}{\not{d}\mathrm{iv}}$ is the divergence operator on $\mathbb{S}^2$ and $\overset{\circ}{\not{d}}$ the differential of a function on $\mathbb{S}^2$. This equation results by taking the limit of the Codazzi equation on $S_{t,u}$

associated to χ:

$$\not{d}\mathrm{iv}\,\hat{\chi} + \hat{\chi}\cdot\zeta = \frac{1}{2}\left(\not{d}\mathrm{tr}\chi + \mathrm{tr}\chi\zeta\right) - \beta \tag{5}$$

Here $\not{d}\mathrm{iv}$ is the divergence operator on $S_{t,u}$, $\not{d}$ the differential of a function on $S_{t,u}$ and β is a component of the spacetime curvature. The contribution of the curvature component vanishes in the limit $t \to \infty$. The torsion ζ of $S_{t,u}$ satisfies an elliptic Hodge system. One equation of the system expresses the curvature of the normal bundle $\mathrm{c}\not{u}\mathrm{rl}\,\zeta$, $\mathrm{c}\not{u}\mathrm{rl}$ being the curl operator on $S_{t,u}$, as:

$$\mathrm{c}\not{u}\mathrm{rl}\,\zeta = \sigma - \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} \tag{6}$$

Here σ is the component of the spacetime curvature defined by the condition that the restriction of $(1/2)R(\cdot,\cdot,\underline{L},L)$ $\sigma\not{\epsilon} = (1/2)R(\cdot,\cdot,\underline{L},L)$ where $\not{\epsilon}$ is the area 2-form of $S_{t,u}$.

The other equation of the Hodge system is expressed in terms of the mass aspect function $\underline{\mu}$ defined by:

$$\underline{\mu} = K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + \not{d}\mathrm{iv}\,\zeta \tag{7}$$

where K is the Gauss curvature of $S_{t,u}$. The mass aspect function plays an important role in the analysis of the Einstein equations. While it is an order 2 quantity like the spacetime curvature, its variation along $\underline{L}$ is by virtue of the Einstein equations again an order 2 quantity rather than an order 3 quantity. By the Gauss-Bonnet formula,

$$\int_{S_{t,u}} \underline{\mu} d\mu_{\not{g}} = \frac{8\pi m}{r} \tag{8}$$

where m, defined by:

$$m(t,u) = \frac{r(t,u)}{2}\left[1 + \frac{1}{16\pi}\int_{S_{t,u}} \mathrm{tr}\chi\mathrm{tr}\underline{\chi} d\mu_{\not{g}}\right] \tag{9}$$

is known as the Hawking mass of $S_{t,u}$. In the above, $d\mu_{\not{g}}$ is the area element of $S_{t,u}$. The limit as $t \to \infty$ of $m(t,u)$, denoted by $M(u)$ is the Bondi mass (more properly, energy) at retarded time u. In fact, as $t \to \infty$, $r^3 \underline{\mu}$ tends to a limit $\underline{N}$ a function on $\mathbb{S}^2$ depending on u, thus another quantity at future null infinity, and we have

$$\overline{\underline{N}} = 2M \tag{10}$$

the overbar denoting mean value on $\mathbb{S}^2$. The equation expressing the variation of $\underline{\mu}$ along $\underline{L}$ gives rise in the limit $t \to \infty$ to the equation

$$\frac{\partial \underline{N}}{\partial u} = -\frac{1}{4}|\Xi|^2 \tag{11}$$

another evolution equation at future null infinity. Taking the mean value of this equation on $\mathbb{S}^2$ we recover the Bondi mass loss formula [B-B-M]:

$$\frac{\partial M}{\partial u} = -\frac{1}{32\pi}\int_{\mathbb{S}^2} |\Xi|^2 d\mu_{\overset{\circ}{\not g}} \tag{12}$$

Thus $|\Xi|^2/32\pi$ is the energy radiated per unit time per unit solid angle at a given retarded time in a given direction.

Now, the Gauss equation of the embedding of $S_{t,u}$ in spacetime reads:

$$K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} = -\rho \tag{13}$$

where $\rho = (1/4)R(\underline{L}, L, \underline{L}, L)$. Substituting for K from (13) into (7) yields the other equation of the Hodge system on $S_{t,u}$:

$$\not{\mathrm{d}}\mathrm{iv}\,\zeta = \underline{\mu} + \rho - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} \tag{14}$$

As $t \to \infty$, $r^3\rho$ tends to a limit P and $r^3\sigma$ tends to a limit Q, both functions on $\mathbb{S}^2$ depending on u. The Hodge system on $S_{t,u}$ (7), (14) gives rise in the limit $t \to \infty$ to the following Hodge system on $\mathbb{S}^2$:

$$\overset{\circ}{\mathrm{c}\not{u}\mathrm{rl}}\ Z = Q - \frac{1}{2}\Sigma\wedge\Xi, \qquad \overset{\circ}{\not{\mathrm{d}}\mathrm{iv}}\ Z = \underline{N} + P - \frac{1}{2}\Sigma\cdot\Xi \tag{15}$$

These imply, taking mean values on $\mathbb{S}^2$,

$$\overline{Q} = \frac{1}{2}\overline{\Sigma\wedge\Xi}, \qquad \overline{P} = -\overline{\underline{N}} + \frac{1}{2}\overline{\Sigma\cdot\Xi}$$

The quantities H and Σ depend on the choice of the surface $S_{0,0}$ in the construction previously discussed. However Ξ is independent of this choice and is an intrinsic property of future null infinity. In view of (1) so is the difference of Σ at the same point of $\mathbb{S}^2$ and two values of u. It turns out that $\Xi = O(|u|^{-3/2})$ as $|u| \to \infty$. It follows Σ tends to limits Σ^- and Σ^+ as $u \to -\infty$ and $u \to +\infty$ respectively and $\Sigma^+ - \Sigma^- = -\frac{1}{2}\int_{-\infty}^{+\infty} \Xi du$. Let us define on $\mathbb{S}^2$ the function

$$F = \frac{1}{8}\int_{-\infty}^{+\infty} |\Xi|^2 du \tag{16}$$

According to the above $F/4\pi$ is the total energy radiated per unit solid angle in a given direction. In view of (11), $\underline{N}$ tends to limits $\underline{N}^-$ and $\underline{N}^+$ as $u \to -\infty$ and $u \to +\infty$ respectively and $\underline{N}^+ - \underline{N}^- = -2F$. It turns out that $Q - \overline{Q} \to 0$ as $|u| \to \infty$. On the other hand $P - \overline{P}$ tends to limits $(P - \overline{P})^-$ and $(P - \overline{P})^+$ as $u \to -\infty$ and $u \to +\infty$. We have $(P - \overline{P})^+ = (P - \overline{P})^- = 0$ if and only if the final center of mass frame is at rest relative to the initial center of mass frame and the initial and final velocities of the masses in the corresponding frames are negligible. This is the case in the context of the problem of the stability of the Minkowski spacetime, because there all the mass is eventually radiated away. In general $(P - \overline{P})^-$ and $(P - \overline{P}^+)$ are determined from sums of boosted Schwarzschild solutions. For example in the case of two bodies, initially in slow motion, which coalesce to one body of final rest mass M^+ and with final velocity V relative to the initial center of mass frame, we have $(P - \overline{P})^- = 0$ while at $\xi \in \mathbb{S}^2 \subset \mathbb{R}^3$:

$$(P - \overline{P})^+(\xi) = -2M^+ \left[\frac{(1 - |V|^2)^{3/2}}{(1 - <\xi, V>)^3} - \frac{1}{(1 - |V|^2)^{1/2}}\right]$$

Making use of the above results we conclude from the system (15) that Z tends to limits Z^- and Z^+ as $u \to -\infty$ and $u \to +\infty$ respectively and $Z^+ - Z^-$ satisfies the Hodge system:

$$\overset{\circ}{\not{\text{curl}}}\ (Z^+ - Z^-) = 0,$$
$$\overset{\circ}{\not{\text{div}}}\ (Z^+ - Z^-) = (P - \overline{P})^+ - (P - \overline{P})^- + \underline{N}^+ - \underline{N}^- - \overline{\underline{N}}^+ + \overline{\underline{N}}^-$$

Hence, defining the function Φ to be the solution, of vanishing mean, of the equation

$$\overset{\circ}{\not\triangle}\,\Phi = (P-\overline{P})^{+} - (P-\overline{P})^{-} - 2(F-\overline{F}) \tag{17}$$

we have:

$$Z^{+} - Z^{-} = \overset{\circ}{\not d}\,\Phi \tag{18}$$

while by (4), in view of (2), the difference $\Sigma^{+}-\Sigma^{-}$ satisfies the equation

$$\overset{\circ}{\not d}\mathrm{iv}\ (\Sigma^{+}-\Sigma^{-}) \tag{19}$$

Equations (17) - (19) determine $\Sigma^{+}-\Sigma^{-}$ uniquely. Their integrability condition is that Φ has vanishing projection $\Phi_{(1)}$ on the 1st eigenspace of $\overset{\circ}{\not\triangle}$. Thus,

$$P^{+}_{(1)} - P^{-}_{(1)} - 2F_{(1)} = 0 \tag{20}$$

holds, which expresses the law of conservation of linear momentum. In the example of binary coalescence $P^{-}_{(1)} = 0$, $P^{+}_{(1)} = -6 < \xi, V > M^{+}/(1-|V|^2)^{1/2}$ and (20) states that the recoil momentum is equal and opposite to the momentum carried off by the waves. It is to be noted that by reason of (20) the right hand side of (17) is actually equal to

$$(P-P_{[1]})^{+} - (P-P_{[1]})^{-} - 2(F-F_{[1]})$$

where the subscript $[1]$ denotes projection on the sum of the 0th and 1st eigenspaces of $\overset{\circ}{\not\triangle}$, the projection on the 0th eigenspace being the mean value. Also that in the example of binary coalescence $(P-P_{[1]})^{+}$ being quadratic and higher in V is quadratic and higher in F.

By the spring of 1990 I had already reached the main conclusion contained in the above results, namely that $\Sigma^{+}-\Sigma^{-}$ is a nonlinear invariant at future null infinity associated to foliations by outgoing null hypersurfaces related by translations at infinity and determined by a physical quantity, the total energy radiated per unit solid angle in a given direction. However I had not yet realized the experimental implications of this conclusion. At that time I was a professor at the Courant Institute in New York but my house was at Berkeley Heights in New Jersey, near the Bell Laboratories at Murray Hill, and I was commuting daily by train to New York. One evening on returning home from New York I was reading on the train an article on the effort to detect gravitational waves using laser interferometers. The article was contained in an issue of a NSF magazine on the progress of various NSF supported projects. One statement in the article particularly caught my attention: that not only must a distance of kilometers be measured with incredible accuracy, less than the diameter of an atomic nucleus, but also that this must be done within a fraction of a second because after each half-period of revolution of an astronomical binary the displacements of the test masses of the interferometer, caused by the emitted gravitational waves, would cancel. On reading this the thought struck me that the nonlinear invariant which I had found should cause a permanent displacement of the test masses of the detector, unknown to the authors of the article.

On returning home that evening and in the ensuing period I worked out the details. A laser interferometer gravitational wave detector essentially consists of a reference mass m_0, which is the beam splitter/synthesizer, and two test masses m_1 and m_2, which carry the reflectors, all initially at rest on a plane, with m_1 and m_2 at equal distances d_0 and at right angles from m_0. When the experiment is performed on the Earth's surface the masses are suspended by pendulums, however for time intervals much shorter than the period of the pendulums the motion of the masses can be considered free. Any difference in the light travel times between m_0 and m_1 and m_0 and m_2 results in a difference of phase of the laser light at m_0. Let τ be the time scale over which the spacetime curvature varies significantly. If the ratio d_0/τ is assumed small, differences in light travel time accurately reflect differences in distance. Defining a parallel propagated orthonormal frame (T, E_1, E_2, E_3) along the world line of m_0 where T is the unit tangent to this world line and E_1, E_2 point initially in the directions of m_1, m_2 respectively, the instantaneous positions of m_1, m_2 relative to m_0 are described in normal coordinates $x^i : i = 1, 2, 3$ based on the world line of m_0. The assumption that d_0/τ is small then implies that the motion of the m_1, m_2 is governed by the Jacobi equation $\ddot{x}^i = -R(E_i, T, E_j, T)x^j$, where $R(E_i, T, E_j, T)(u)$ are the curvature components in the orthonormal frame along the world line of m_0 and we denote by a dot differentiation with respect to u. Let us denote by $x^i_{(A)}$ the ith coordinate of the mass $m_A : A = 1, 2$. The initial conditions are $x^i_{(A)}(-\infty) = d_0\delta^i_A$, $\dot{x}^i_{(A)}(-\infty) = 0$. Suppose for simplicity that the source is in the direction of E_3, namely the vertical. Then horizontal plane is identified with the tangent plane to $\mathbb{S}^2$ at the point corresponding to the direction from the source, and to the vectors (E_1, E_2) there corresponds an orthonormal frame for $\mathbb{S}^2$ at that point. Then, denoting by $\underline{A}_{AB} : A, B = 1, 2$ the components of $\underline{A}$ in this frame (see (3)) the results on the asymptotic behavior of the space-time curvature components (see paragraph above (3)) imply that $\ddot{x}^3_{(A)} = O(r^{-3})$, $\ddot{x}^A_{(B)} = -(1/4)r^{-1}\underline{A}_{AC}x^C_{(B)} + O(r^{-2})$. Since r^{-2} is negligible in comparison with r^{-1}, this means that the motion of the test masses is essentially confined to the horizontal plane and in view of the fact that the displacements displacements are negligible in comparison to d_0, we can replace $x^C_{(B)}$ in the 2nd equation by its initial value $d_0\delta^C_B$ obtaining $\ddot{x}^A_{(B)} = -(d_0/4r)\underline{A}_{AB}$. Integrating once then gives, by virtue of (3), $\dot{x}^A_B = (d_0/2r)\Xi_{AB}$. Integrating again we obtain, by virtue of (1),

$$x^A_{(B)}(u) - x^A_{(B)}(-\infty) = -(d_0/2r)(\Sigma_{AB}(u) - \Sigma^-_{AB})$$

Taking the limit $u \to \infty$ we conclude that the test masses suffer permanent displacements given by:

$$\triangle x^A_{(B)} = -(d_0/r)(\Sigma^+_{AB} - \Sigma^-_{AB}) \tag{21}$$

I submitted the paper to Phys. Rev. Letters in December 1990. I was very fortunate that it reached Kip Thorne, who invited me to Caltech to give talks. He was very supportive and wrote a follow-up article [T] which was instrumental in the nonlinear memory effect receiving the attention of the physics, in particular the experimental physics, community. While at that time the experimental verification of the effect seemed a long way off in the future, today, after the tremendous success of the LIGO/Virgo project in detecting gravitational waves from inspiring binary black holes and neutron stars, this prospect does not seem so distant any more. See [L-T-L-B-C].

I now turn to what was chronologically the last of the works for which I was awarded a Henri Poincaré Prize, as it belongs to the same framework as the work just discussed. This is the formation of black holes in pure general relativity by the focusing of incoming gravitational waves. The problem had been proposed to me by my teacher, John Wheeler, in the summer of 1968, just before I entered graduate school, as a distant goal. As I was completing the work [C3] which contains the solution of the problem, some 40 years later, I learned that John Wheeler had passed away a few weeks earlier.

The concept of a trapped surface was introduced by Penrose in 1965 [P]. He defined a trapped surface to be a closed spacelike surface in spacetime, such that an infinitesimal virtual displacement of the surface along either family of future-directed null geodesic normals to the surface, the incoming family as well as the outgoing family, leads to a pointwise decrease of the area element. On the basis of this concept, Penrose proved a remarkable theorem, which states that a spacetime arising from a non-compact Cauchy hypersurface and containing a trapped surface must be future null geodesically incomplete. The term “black hole” was coined by John Wheeler in 1967. Shortly afterward a black hole was defined mathematically as the complement in spacetime of the past of future null infinity. It was also shown that any trapped surface must lie in this complement, therefore the presence of a trapped surface implies the presence of a black hole containing it.

Evidently, the incompleteness theorem of Penrose presupposes the presence of a trapped surface. The challenge posed by Wheeler’s problem was then to develop methods of analysis, in the context of the Einstein vacuum equations, which, even when the initial conditions are arbitrarily far from already containing trapped surfaces, allow us to follow the long time evolution and show that, under suitable circumstances, trapped surfaces eventually form.

The simplest to state version of the theorem which [C3] establishes is the limiting version, where we have an asymptotic characteristic initial value problem with initial data at past null infinity. Denoting by $\underline{u}$ the “advanced time”, it is assumed that the initial data are trivial for $\underline{u} \leq 0$.

Let k, l be positive constants, $k > 1$, $l < 1$. Let us be given smooth asymptotic initial data at past null infinity which is trivial for advanced time $\underline{u} \leq 0$. Suppose that the incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$ is not less than $k/8\pi$. Then if δ is suitably small, the maximal development of the data contains a trapped surface S which is diffeomorphic to $\mathbb{S}^2$ and has area Area$(S) \geq 4\pi l^2$.

The above theorem is obtained through a theorem in which the initial data is given on a complete future null geodesic cone C_o. The generators of the cone are parametrized by an affine parameter s measured from the vertex o and defined so that the corresponding null geodesic vector field has projection T at o along a fixed unit future-directed timelike vector T at o. It is assumed that the initial data are trivial for $s \leq r_0$, for some $r_0 > 1$. The boundary of this trivial region is then a round sphere of radius r_0. The advanced time $\underline{u}$ is then defined along C_o by $\underline{u} = s - r_0$. The formation of trapped surfaces theorem is similar in this case, the only difference being that the "incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$", a notion defined only at past null infinity, is replaced by the integral

$$\frac{r_0^2}{8\pi} \int_0^\delta e d\underline{u}$$

on the affine parameter segment $[r_0, r_0 + \delta]$ of each generator of C_o. The function e is an invariant of the conformal intrinsic geometry of C_o, given by:

$$e = \frac{1}{2} |\hat{\chi}|^2_{\not{g}}$$

where $\not{g}$ is the induced metric on the sections of C_o corresponding to constant values of the affine parameter, and, as the preceding discussion, $\hat{\chi}$ is the trace-free part of their 2nd fundamental form relative to C_o.

The theorem for a cone C_o is established for any $r_0 > 1$ and the smallness condition on δ is independent of r_0. The domain of dependence, in the maximal development, of the trivial region in C_o is a domain in Minkowski spacetime bounded in the past by the trivial part of C_o and in the future by $\underline{C}_e$, the past null geodesic cone of a point e at arc length $2r_0$ along the timelike geodesic Γ_0 from o with tangent vector T at o.

Almost all the work is in fact devoted to establishing an existence theorem for a development of the initial data which extends far enough into the future so that trapped spheres have eventually a chance to form within this development. This existence theorem gives us full knowledge of the geometry of spacetime when trapped surfaces begin to form.

The new method which the work [C3] introduces and which enables the solution of the problem is the short pulse method. This is a method of treating the focusing of incoming waves, and it is of wider applicability, beyond the domain of general relativity. The method depends on the assumption that the initial data displays an abrupt change across a smooth surface, hence there is a small length in the problem, representing the distance from the surface within which the change is accomplished. The short pulse method allows us to establish an existence theorem for a development of the initial data which is large enough so that interesting things have a chance to occur within this development. The Einstein vacuum equations being scale invariant, one may ask what does it mean for a length to be small in this context. Here small means by comparison to the area radius of the trapped sphere to be formed.

The short pulse method came to me in a most unexpected way. It was the spring of 2004 and I was visiting New York. I had not thought about the problem for more than a decade. In fact, for the preceding five years I had been working on the formation of shocks and I had reached an impasse at that time. Thus my mind was fully occupied with overcoming this impasse. That day I took a trip by train to Princeton to see friends. On returning in the evening to my hotel in New York I fell into a deep sleep for several hours. I then came out of the deep sleep but did not wake up. I was rather suspended in a state between deep sleep and awakedness, where, in the absence of external stimuli, maximum concentration is possible. It was in this state that the short pulse method came to me. The next morning I wrote down a few pages of notes and more in the next few days after returning to Europe. However I postponed working out the details until after I had completed my work [C2] on the formation of shocks, which took two more years. After another two years of concentrated work the monograph [C3] was complete.

Here we are considering incoming waves and it is the incoming null hypersurfaces $\underline{C}_{\underline{u}}$, the level sets in spacetime of the advanced time $\underline{u}$, which follow these waves. With initial data on the complete future null geodesic cone C_o which are trivial for $s \leq r_0$ as discussed above, we consider the restriction of the initial data to $s \leq r_0 + \delta$. In terms of the advanced time $\underline{u}$, we restrict attention to the interval $[0, \delta]$, the data being trivial for $\underline{u} \leq 0$. We define the "retarded time" u by the condition that its level sets C_u are, like the initial hypersurface C_o itself, future null geodesic cones with vertices on the timelike geodesic Γ_0, together with the condition that, with $u_0 = -r_0$, $u - u_0$ is along Γ_0 one half the arc length from o. Then C_o may also be denoted by C_{u_0}. The development whose existence we want to establish is that bounded in the future by the spacelike hypersurface H_{-1} where $\underline{u} + u = -1$ and by the incoming null hypersurface $\underline{C}_\delta$. We denote this development M_{-1}. We confine attention to the nontrivial region $\underline{u} \geq 0$ of M_{-1}. We denote by $S_{\underline{u},u}$ the surfaces of intersection $S_{\underline{u},u} = \underline{C}_{\underline{u}} \bigcap C_u$ which are spacelike and diffeomorphic to $\mathbb{S}^2$. They play the role of the surfaces $S_{t,u}$ of the preceding discussion. What is proved in the trapped surface formation theorem is that $S_{\delta,-1-\delta}$, the outermost sphere on the future boundary of M_{-1}, is trapped.

We define L and $\underline{L}$ to be the future directed null vector fields the integral curves of which are the generators of the C_u and $\underline{C}_{\underline{u}}$, parametrized by $\underline{u}$ and u respectively, so that

$$Lu = \underline{L}\underline{u} = 0, \quad L\underline{u} = \underline{L}u = 1$$

The flow Φ_τ generated by L defines a diffeomorphism of $S_{\underline{u},u}$ onto $S_{\underline{u}+\tau,u}$, while the flow $\underline{\Phi}_\tau$ generated by $\underline{L}$ defines a diffeomorphism of $S_{\underline{u},u}$ onto $S_{\underline{u},u+\tau}$. The two flows do not commute. The positive function Ω defined by $g(L,\underline{L}) = -2\Omega^2$ may be thought of as the inverse density of the double null foliation. The optical quantities are the geometric quantities defined by the double null foliation. These are the induced metric $\not{g}$ on the surfaces $S_{\underline{u},u}$, the associated Gauss curvature K, the function Ω, the two 2nd fundamental forms χ and $\underline{\chi}$, associated to $S_{\underline{u},u}$ and to L and $\underline{L}$ respectively, the torsion ζ of $S_{\underline{u},u}$, $\not{d}\Omega$ and $L\Omega$, $\underline{L}\Omega$. The commutator $[\underline{L}, L]$, which determines the non-commutativity of the two flows, is expressed in terms of ζ.

The optical structure equations are the equations satisfied by the optical quantities. These equations involve the spacetime curvature components. Definining the normalized null normal vector fields $\hat{L} = \Omega^{-1}L$, $\hat{\underline{L}} = \Omega^{-1}\underline{L}$ so that $g(\hat{L},\hat{\underline{L}}) = -2$, these components are the trace-free symmetric 2-covariant tensor fields α and $\underline{\alpha}$ on the $S_{\underline{u},u}$ which represent the restrictions to $S_{\underline{u},u}$ of the $R(\cdot,\hat{L},\cdot,\hat{L})$ and $R(\cdot,\hat{\underline{L}},\cdot,\hat{\underline{L}})$ parts of the spacetime curvature respectively, the 1-forms β and $\underline{\beta}$ on the $S_{\underline{u},u}$ which represent the restrictions to $S_{\underline{u},u}$ of the $(1/2)R(\cdot,\hat{L},\hat{\underline{L}},\hat{L})$ and $(1/2)R(\cdot,\hat{\underline{L}},\hat{\underline{L}},\hat{L})$ parts of the spacetime curvature respectively, and the functions ρ and σ defined by $\rho = (1/4)R(\hat{\underline{L}},\hat{L},\hat{\underline{L}},\hat{L})$ and by the condition that $\sigma\notin$ is the restriction to $S_{\underline{u},u}$ of the $(1/2)R(\cdot,\cdot,\hat{\underline{L}},\hat{L})$ part of the curvature. Here $\notin$ is the area form of $S_{\underline{u},u}$.

The first step in the short pulse method is the analysis of the equations along the initial hypersurface C_{u_0}. The analysis is particularly clear and simple because of the fact that C_{u_0} is a null hypersurface, so we are dealing with the characteristic initial value problem and there is a way of formulating the problem in terms of free data which are not subject to any constraints. The full set of data which includes all the curvature components and their transversal derivatives up to any given order along C_{u_0}, is then determined by integrating ordinary differential equations along the generators of C_{u_0}. The free data is the conformal intrinsic geometry of C_{u_0}. This may be described as a 2-covariant symmetric positive definite tensor density m, of weight -1 and unit determinant, on $\mathbb{S}^2$, depending on $\underline{u}$. This is of the form:

$$m = \exp\psi$$

where ψ is a 2-dimensional symmetric trace-free matrix valued function on $\mathbb{S}^2$, depending on $\underline{u} \in [0,\delta]$, and transforming under change of charts on $\mathbb{S}^2$ in such a way so as to make m a 2-covariant tensor density of weight -1. The transformation rule is particularly simple if stereographic charts on $\mathbb{S}^2$ are used. Then there is a function O defined on the intersection of the domains of the north and south polar stereographic charts on $\mathbb{S}^2$, with values in the 2-dimensional symmetric orthogonal matrices of determinant -1 such that in going from the north polar chart to the south polar chart or vise-versa, $\psi \mapsto \tilde{O}\psi O$ and $m \mapsto \tilde{O}mO$.

The crucial ansatz of the short pulse method is the following. We consider an arbitrary smooth 2-dimensional symmetric trace-free matrix valued function ψ_0 on $\mathbb{S}^2$, depending on $s \in [0,1]$, which extends smoothly by 0 to $s \leq 0$, the "seed data", and we set:

$$\psi(\underline{u},\vartheta) = \frac{\delta^{1/2}}{|u_0|}\psi_0\left(\frac{\underline{u}}{\delta},\vartheta\right), \quad (\underline{u},\vartheta) \in [0,\delta]\times\mathbb{S}^2 \tag{22}$$

The analysis of the equations along C_{u_0} then gives, for the components of the spacetime curvature along C_{u_0}:

$$\sup_{C_{u_0}} |\alpha| \leq O_2(\delta^{-3/2}|u_0|^{-1}), \quad \sup_{C_{u_0}} |\beta| \leq O_2(\delta^{-1/2}|u_0|^{-2})$$
$$\sup_{C_{u_0}} |\rho|, \ \sup_{C_{u_0}} |\sigma| \leq O_3(|u_0|^{-3})$$
$$\sup_{C_{u_0}} |\underline{\beta}| \leq O_4(\delta|u_0|^{-4}), \quad \sup_{C_{u_0}} |\underline{\alpha}| \leq O_5(\delta^{3/2}|u_0|^{-5}) \tag{23}$$

Here, the symbol $O_k(\delta^p|u_0|^r)$ means the product of $\delta^p|u_0|^r$ with a non-negative non-decreasing continuous function of the C^k norm of ψ_0 on $[0,1] \times \mathbb{S}^2$. The pointwise magnitudes of tensors on $S_{\underline{u},u}$ are with respect to the induced metric $\not{g}$, which is positive definite, the surfaces being spacelike.

One should focus on the dependence on δ of the right hand sides of (23). This displays what we may call the short pulse hierarchy. And this hierarchy is nonlinear. For, if only the linearized form of the equations was considered, a different hierarchy would be obtained: the exponents of δ in the first two of (23) would be the same, but the exponents of δ in the last three of (23) would instead be $1/2, 3/2, 5/2$, respectively. The short pulse hierarchy is the key to the existence theorem as well as to the trapped surface formation theorem.

The main step is of course demonstrating that the short pulse hierarchy is preserved in evolution. This is accomplished though energy-type estimates for the spacetime curvature in conjunction with estimates for the optical quantities in terms of the spacetime curvature derived by analyzing the optical structure equations, an approach which originated in [C-K]. A refinement of this approach in accordance with the short pulse hierarchy is used here.

A Weyl field on a 4-dimensional Lorentzian manifold is a 4-covariant tensor field with the algebraic properties of the Weyl, or conformal, curvature tensor. A Weyl field W is subject to equations which are analogues of Maxwell's equations for the electromagnetic field. They are of the form:

$$\nabla^\alpha W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta}$$

and we call the right hand side J a "Weyl current". The fundamental Weyl field is the spacetime curvature, in view of the fact that the vacuum Einstein equations express the vanishing of the Ricci curvature. In this case the associated Weyl current vanishes and the Bianchi equations reduce to the Bianchi identities. Given a vector field Y and a Weyl field W or Weyl current J, there is a modified Lie derivative of W, J with respect to Y, denoted $\tilde{\mathcal{L}}_Y W$, $\tilde{\mathcal{L}}_Y J$, which is also a Weyl field or Weyl current respectively. The conformal covariance properties of the Bianchi equations imply the following. If J is the Weyl current associated to the Weyl field W according to the Bianchi equations, then the Weyl current associated to $\tilde{\mathcal{L}}_Y W$ is the sum of $\tilde{\mathcal{L}}_Y J$ and a bilinear expression which is on one hand linear in ${}^{(Y)}\tilde{\pi}$ and its first covariant derivative and on the other hand in W and its first covariant derivative. Here we denote by ${}^{(Y)}\tilde{\pi}$ the deformation tensor of Y, namely the trace-free part of the Lie derivative of the metric g with respect to Y. This measures the rate of change of the conformal spacetime geometry under the flow generated by Y. From the fundamental Weyl field and a set of vector fields $Y_1, ..., Y_n$ which we call "commutation fields", derived Weyl fields of up to any given order m are generated by the repeated application of the operators $\tilde{\mathcal{L}}_{Y_i} : i = 1, ..., n$. A basic requirement on the set of commutation fields is that it spans the tangent space to M at each point. The Weyl currents associated to these derived Weyl fields are then determined by the deformation tensors of the commutation fields.

Here, as commutation fields we take L, S defined by:

$$S = u\underline{L} + \underline{u}L$$

and the three rotation fields $O_i \ : i = 1, 2, 3$. The latter are defined as follows. In the Minkowskian region we introduce rectangular coordinates $x^\mu \ : \mu = 0, 1, 2, 3$, taking the x^0 axis to be the timelike geodesic Γ_0. In the Minkowskian region, in particular on the sphere S_{0,u_0}, the O_i are the generators of rotations about the $x^i \ : i = 1, 2, 3$ spatial coordinate axes. The O_i are then first defined on C_{u_0} by conjugation with the flow of L and then in spacetime by conjugation with the flow of $\underline{L}$. The deformation tensors of the commutation fields are then determined in terms of the optical quantities. The Weyl fields which we consider are, besides the fundamental Weyl field R, the spacetime curvature, the following derived Weyl fields

1st order: $\tilde{\mathcal{L}}_L R,\ \tilde{\mathcal{L}}_{O_i} R : i = 1, 2, 3,\ \tilde{\mathcal{L}}_S R$

2nd order: $\tilde{\mathcal{L}}_L \tilde{\mathcal{L}}_L R,\ \tilde{\mathcal{L}}_{O_i} \tilde{\mathcal{L}}_L R : i = 1, 2, 3,\ \tilde{\mathcal{L}}_{O_j} \tilde{\mathcal{L}}_{O_i} R : i, j = 1, 2, 3,$

$\tilde{\mathcal{L}}_{O_i} \tilde{\mathcal{L}}_S R : i = 1, 2, 3,\ \tilde{\mathcal{L}}_S \tilde{\mathcal{L}}_S R$

We assign to each Weyl field the index l according to the number of $\tilde{\mathcal{L}}_L$ operators in the definition of W in terms of R.

Given a Weyl field W there is a 4-covariant tensor field $Q(W)$ associated to W, the Bel-Robinson tensor, which is totally symmetric and trace-free in any pair of indices. It is a quadratic expression in W, analogous to the Maxwell energy-monentum-stress tensor for the electromagnetic field. The Bel-Robinson tensor has a remarkable positivity property:$Q(W)(X_1, X_2, X_3, X_4)$ is non-negative for any tetrad X_1, X_2, X_3, X_4 of future directed non-spacelike vectors at a point. Moreover, the divergence of $Q(W)$ is a bilinear expression which is linear in W and in the associated Weyl current J. Given a Weyl field W and a triplet of future directed non-spacelike vector fields X_1, X_2, X_3, which we call "multiplier fields" we define the energy-momentum density vector field $P(W; X_1, X_2, X_3)$ associated to W and to the triplet X_1, X_2, X_3 by:

$$P(W; X_1, X_2, X_3)^\alpha = -Q(W)^\alpha_{\beta\gamma\delta} X_1^\beta X_2^\gamma X_3^\delta$$

Then the divergence of $P(W; X_1, X_2, X_3)$ is the sum of $-(\text{div}Q(W))(X_1, X_2, X_3)$ and a bilinear expression which is linear in $Q(W)$ and in the deformation tensors of X_1, X_2, X_3. The divergence theorem in spacetime, applied to a domain which is a development of part of the initial hypersurface, then expresses the integral of the 3-form dual to $P(W; X_1, X_2, X_3)$ on the future boundary of this domain, in terms of the integral of the same 3-form on the past boundary of the domain, namely on the part of the initial hypersurface, and the spacetime integral of the divergence. The boundaries being achronal - that is, no pair of points on each boundary can be joined by a timelike curve - the integrals are integrals of non-negative functions, by virtue of the positivity property of $Q(W)$.

Here as multiplier fields we take the vector fields L and K, where

$$K = u^2 \underline{L}$$

The deformation tensors of the multiplier fields are then determined in terms of the optical quantities. For each of the above defined Weyl fields we define the energy-momentum density vector fields

$$\overset{(n)}{P}(W) \ : \ n = 0, 1, 2, 3$$

where:

$$\overset{(0)}{P}(W) = P(W; L, L, L) \quad \overset{(1)}{P}(W) = P(W; K, L, L)$$
$$\overset{(2)}{P}(W) = P(W; K, K, L) \quad \overset{(3)}{P}(W) = P(W; K, K, K)$$

We then define the total 2nd order energies $\overset{(n)}{E}_2(u)$ as the integrals on the C_u and the total 2nd order fluxes $\overset{(n)}{F}_2(\underline{u})$ as the integrals on the $\underline{C}_{\underline{u}}$, of the 3-forms dual to the $\overset{(n)}{P}_2$. Of the fluxes only $\overset{(3)}{F}_2(\underline{u})$ plays a role in the problem. Finally, with the exponents $q_n \ : \ n = 0, 1, 2, 3$ defined by:

$$q_0 = 1, \ \ q_1 = 0, \ \ q_2 = -\frac{1}{2}, \ \ q_3 = -\frac{3}{2},$$

according to the short pulse hierarchy, we define the quantities

$$\overset{(n)}{\mathcal{E}}_2 = \sup_u \left(\delta^{2q_n} \overset{(n)}{E}_2(u) \right) \ : n = 0, 1, 2, 3; \ \ \overset{(3)}{\mathcal{F}}_2 = \sup_{\underline{u}} \left(\delta^{2q_3} \overset{(3)}{F}_2(\underline{u}) \right) \tag{24}$$

The objective then is to obtain bounds for these quantities in terms of the initial data.

In view of the fact that by virtue of the above definitions the deformations tensors of the multiplier fields and the commutation fields are determined in terms of the optical quantities, this reduces to appropriately estimating the optical quantities in terms of the spacetime curvature by analyzing the optical structure equations.

Now, the estimates of the error integrals, namely the integrals of the absolute values of the divergences of the $\overset{(n)}{P}_2$, yield inequalities for the quantities (24). These inequalities contain, besides the initial data terms

$$\overset{(n)}{D} = \delta^{2q_n} \overset{(n)}{E}_2(u_0) \ : \ n = 0, 1, 2, 3,$$

terms of $O(\delta^p)$ for some $p > 0$, which are innocuous, as they can be made less than or equal to 1 by subjecting δ to a suitable smallness condition, but they also contain terms of $O(1)$ which are nonlinear in the quantities (24). From such a nonlinear system of inequalities, no bounds can in general be deduced, because here, in contrast to [C-K], the initial data quantities are allowed to be arbitrarily large. However a fortunate circumstance occurs: our system of inequalities is "reductive". That is, the inequalities, taken in proper sequence, reduce to a sequence of sublinear inequalities, thus allowing us to obtain the sought for bounds.

I shall now discuss the remaining work for which I was awarded a Henri Poincaré Prize, namely the work [C2] on the formation of shocks in fluids. This work is in the context of the relativistic Euler equations. The reason why I addressed the relativistic problem first was that I expected that in the relativistic theory, because of its unified view of space and time, geometric spacetime structures would be more readily apparent. The mechanics of a perfect fluid are described in the framework of special relativity by a future-directed timelike vector field u of unit magnitude relative to the Minkowski metric g, the fluid 4-velocity, and two positive functions n and s, the number of particles per unit volume and the entropy per particle. The mechanical properties of a perfect fluid are determined once we give an equation of state, which expresses the mass-energy density ρ as a function of n and s:

$$\rho = \rho(n, s)$$

According to the laws of thermodynamics, the pressure p and the temperature θ are then given by:

$$p = n\frac{\partial \rho}{\partial n} - \rho, \qquad \theta = \frac{1}{n}\frac{\partial \rho}{\partial s}$$

The particle current is the vector field I given by:

$$I^\mu = nu^\mu$$

The energy-momentum-stress tensor is the symmetric 2-contravariant tensor field T given by:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p(g^{-1})^{\mu\nu}$$

The equations of motion are the differential conservation laws:

$$\nabla_\mu I^\mu = 0, \qquad \nabla_\nu T^{\mu\nu} = 0 \tag{25}$$

Let us define the vorticity 2-form ω by:

$$\omega = -d\beta \tag{26}$$

where β is the 1-form:

$$\beta_\mu = -\sqrt{\sigma} u_\mu, \qquad u_\mu = g_{\mu\nu} u^\nu \tag{27}$$

$\sqrt{\sigma}$ being the relativistic enthalpy per particle:

$$\sqrt{\sigma} = \frac{\rho + p}{n}$$

The differential energy-momentum conservation laws (2nd of (25)) are then seen to be equivalent to the equation:

$$i_u \omega = \theta ds \tag{28}$$

The irrotational case is the case where $\beta = d\phi$ for some function ϕ. In this case (28) implies that s is constant. Only the differential particle conservation (1st of (25)) then remains, which takes the form of a nonlinear wave equation:

$$\nabla_\mu (G \partial^\mu \phi) = 0, \qquad \partial^\mu \phi = (g^{-1})^{\mu\nu} \partial_\nu \phi \tag{29}$$

where

$$G = \frac{n}{\sqrt{\sigma}} = G(\sigma), \qquad \sigma = -(g^{-1})^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Returning to the general case, the sound speed η is defined by:

$$\left(\frac{dp}{d\rho}\right)_s = \eta^2 \tag{30}$$

it being assumed that the left hand side is positive. The acoustical metric h is another Lorentzian metric on the same manifold defined by:

$$h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2) u_\mu u_\nu, \qquad u_\mu = g_{\mu\nu} u^\nu \tag{31}$$

The null cones of h are called "sound cones". The condition $\eta < 1$ is imposed, which means that the sound cones are contained within the null cones of g. What is important is the conformal geometry defined by h, which is equivalent to the acoustical causal structure that is the specification of the causal future and causal past of each point.

Choosing a time function t in Minkowski spacetime, equal to the coordinate x^0 of some rectangular coordinate system, we denote by Σ_t an arbitrary level set of the function t. The Σ_t are parallel spacelike hyperplanes relative to the Minkowski metric g.

Initial data for the equations of motion (25) are given on a domain in the hyperplane Σ_0, which may be the whole of Σ_0. To any given initial data there corresponds a unique maximal classical development of the equations of motion (25), or of the nonlinear wave equation (29) in the irrotational case. The notion of maximal development of given initial data is, in this context, the following. Given initial data, the local existence theorem asserts the existence of a development of this data, namely of a domain $\mathcal{D}$ in Minkowski spacetime, whose past boundary is the domain of the initial data, and of a solution defined in $\mathcal{D}$ and taking the given data at the past boundary, such that the following condition holds. If we consider any point $p \in \mathcal{D}$ and any curve issuing at p with the property that its tangent vector at any point q belongs to the interior or the boundary of the past component of the sound cone at q, then the curve terminates at a point of the domain of the initial data. The local uniqueness theorem asserts that two developments of the same initial data, with domains $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, coincide in $\mathcal{D}_1 \bigcap \mathcal{D}_2$. It follows that the union of all developments of a given initial data is itself a development, the unique maximal development of the initial data.

In [C2] I consider regular initial data on Σ_0 for the general equations of motion (25) which outside a sphere coincide with the data corresponding to a constant state. That is, outside that sphere n and s are constant and u coincides with the future-directed unit normal to Σ_0. I show that under a suitable restriction on the size of the departure of the initial data from those of the constant state, I can control the solution for a time interval of order $1/\eta_0$, where η_0 is the sound speed in the surrounding constant state. I then show that at the end of this time interval a thick annular region has formed, bounded by concentric spheres, where the flow is irrotational, the constant state holding outside the outer sphere. I then study the maximal classical development of the restriction of the data at this time to the exterior of the inner sphere. Thus the global aspects of my work pertain to the irrotational case. I shall therefore confine myself in the following discussion to the case that the initial data are irrotational hence so is the maximal classical development.

Let O be the center of the sphere $S_{0,0}$ in Σ_0 outside which we have the constant state. Let u be a smooth function without critical points on $\Sigma_0 \setminus O$ such that the restriction of u to the exterior of $S_{0,0}$ is equal to minus the Euclidean distance from $S_{0,0}$. We extend u to the spacetime manifold by the condition that its level sets are outgoing null hypersurfaces relative to the acoustical metric h. We call u an "acoustical function" and we denote by C_u an arbitary level set of u. Each C_u is generated by null geodesics of h. Let L be the tangent vector field to these geodesic generators parametrized not affinely but by t. We then define the surfaces $S_{t,u}$ to be $C_u \bigcap \Sigma_t$. Finally we define the vector field T to be tangential to the Σ_t and so that the flow generated by T on each Σ_t is the normal, relative to the induced on Σ_t acoustical metric $\overline{h}$, flow of the foliation of Σ_t by the surfaces $S_{t,u}$. So T is the tangent vector field to the normal curves parametrized by u.

The geometry of a foliation of spacetime by the outgoing acoustically null hypersurfaces C_u, the level sets of u, plays a fundamental role in the problem. The most important geometric property of this foliation from the point of view of the study of shock formation is the density of the packing of its leaves C_u. One measure of this density is the "inverse spatial density", that is, the inverse density of the foliation of each spatial hyperplane Σ_t by the surfaces $S_{t,u}$. This is simply the magnitude κ of the vector field T with respect to h. Another measure is the "inverse temporal density", the function μ, given in arbitrary coordinates by:

$$\frac{1}{\mu} = -(h^{-1})^{\mu\nu}\partial_\mu t \partial_\nu u$$

The two measures are related by:

$$\mu = \alpha\kappa$$

where α is the inverse density, with respect to the acoustical metric h, of the foliation of spacetime by the hyperplanes Σ_t. The function α is bounded above and below by positive constants. Consequently μ and κ are equivalent measures of the density of the packing of the leaves of the foliation of spacetime by the C_u. Shock formation is characterized by the blow up of this density or equivalently by the vanishing of κ or μ.

The domain of the maximal development being a domain in Minkowski spacetime, which by a choice of rectangular coordinates is identified with $\mathbb{R}^4$, inherits the subset topology and the standard differential structure induced by the rectangular coordinates x^α. Choosing an acoustical function u we introduce acoustical coordinates (t, u, ϑ), $\vartheta \in \mathbb{S}^2$, the coordinate lines corresponding to a given value of u and to constant values of ϑ being the generators of C_u. The rectangular coordinates x^α are smooth functions of the acoustical coordinates (t, u, ϑ) and the Jacobian of the transformation is, up to a multiplicative factor which is bounded above and below by positive constants, the function μ. The acoustical coordinates induce another differential structure, the "acoustical differential structure" on the same underlying topological manifold. Since $\mu > 0$ in the interior of the domain of the maximal development, the two differential structures coincide in this interior. The main theorem of [C2] asserts that relative to the acoustical differential structure the maximal classical development extends smoothly to the boundary of its domain. This boundary contains however a singular part B where the function μ vanishes. The rectagular coordinates themselves extend smoothly to the boundary but the Jacobian vanishes at B. As a result, the two differential structures no longer coincide when B is included. The acoustical differential structure admits more smooth functions. With respect to the standard differential structure the solution is continuous but not differentiable at B, the derivative $\hat{T}^\mu \hat{T}^\nu \partial_\mu \partial_\nu \phi$ blowing up as we approach B. Here $\hat{T} = \kappa^{-1} T$, is the vector field of unit magnitude with respect to h corresponding to T. With respect to the standard differential structure, the acoustical metric h is everywhere in the closure of the domain of the maximal development non-degenerate and continuous, but it is not differentiable at B, while with respect to the acoustical differential structure h is everywhere smooth, but it is degenerate at B.

A starting point for the approach leading to the proof of the main theorem is the observation that any variation ψ of ϕ through solutions of the nonlinear wave equation (29) is itself a solution of the linear wave equation

$$\Box_{\tilde{h}} \psi = 0 \tag{32}$$

relative to the conformal acoustical metric $\tilde{h} = \Omega h$, where the conformal factor Ω is the ratio of a function of σ to the value of this function in the surrounding constant state, thus Ω is equal to unity in the constant state. It turns out moreover that Ω is bounded above and below by positive constants.

Here the first order variations correspond to the one-parameter subgroups of the Poincaré group, the isometry group of Minkowski spacetime, extended by the one-parameter scaling or dilation group, which leave the surrounding constant state invariant. The higher order variations are generated from the first order variations by a set of vector fields, the commutation fields. These higher order variations satisfy inhomogeneous wave equations

$$\Box_{\tilde{h}}\psi = \tilde{\rho}$$

the source functions $\tilde{\rho}$ depending on the deformation tensors of the commutation fields. In the present context, for any vector field X in spacetime we call "deformation tensor" of X the Lie derivative of the conformal acoustical metric $\tilde{h}$ with respect to X, denoted by ${}^{(X)}\tilde{\pi}$. The commutation fields are five: the vector field T which is transversal to the C_u, the field $Q = (1+t)L$ along the generators of the C_u and the three rotation fields $R_i \ : i = 1, 2, 3$ which are tangential to the $S_{t,u}$ sections. The latter are defined to be $\Pi \overset{\circ}{R}_i : i = 1, 2, 3$, where the $\overset{\circ}{R}_i \ \ i = 1, 2, 3$ are the generators of spatial rotations associated to the background Minkowskian structure, while Π is the h-orthogonal projection to the $S_{t,u}$.

Two multiplier fields are used:

$$K_0 = (\eta_0^{-1} + \alpha^{-1}\kappa)L + \underline{L}, \qquad \underline{L} = \alpha^{-1}\kappa L + 2T$$

and:

$$K_1 = (\omega/\nu)L$$

Here ν is the mean curvature of the $S_{t,u}$ relative to their null normal L, with respect to the conformal acoustical metric $\tilde{h}$. The function ω is required to have linear growth in t and to be such that $\Box_{\tilde{h}}\omega$ is suitably bounded. The definitions of the multiplier fields and those of the commutation fields are conditioned by the requirement that these vector fields be smooth up to B with respect to the acoustical differential structure. We note that the acoustical differential structure admits fewer smooth vector fields.

To a variation ψ, of any order, and to each of K_0, K_1, an “energy current” is associated, a vector field which is a quadratic expression in $(d\psi,\psi)$. These energy currents define the energies $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, and fluxes $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$. For given t and u the energies are integrals over the exterior of the surface $S_{t,u}$ in the hyperplane Σ_t, while the fluxes are integrals over the part of the outgoing null hypersurface C_u between the hyperplanes Σ_0 and Σ_t. The energies and fluxes are positive-definite by virtue of the fact that the multiplier fields are non-spacelike future-directed relative to the acoustical metric h. It is these energy and flux integrals, together with a spacetime integral $K[\psi](t,u)$ associated to K_1, which are used to control the solution. This spacetime integral is the integral of

$$-\frac{1}{2}(\omega/\nu)(L\mu)_-|\not{d}\psi|^2 \tag{33}$$

in the spacetime region exterior to C_u and bounded by Σ_0 and Σ_t. Minus the function (33) is a negative part of the divergence of the energy current associated to K_1, which is brought to the other side of the corresponding integral identity. In the region where μ is less than a given fraction of η_0, $L\mu$ is bounded from above by a given negative function of t. As a result, the integral $K[\psi](t,u)$ gives effective control of the derivatives of the variations tangential to the $S_{t,u}$ in the region where shocks are to form. The divergence of the energy currents, which determines the growth of the energies and fluxes, depends, besides on the corresponding source function $\tilde{\rho}$, also on ${}^{(K_0)}\tilde{\pi}$, in the case of the energy current associated to K_0, or on ${}^{(K_1)}\tilde{\pi}$, in the case of the energy current associated to K_1.

The deformation tensors of the commutation fields as well as the multiplier fields ultimately depend on the acoustical function u, or, what is the same, on the geometry of the foliation of spacetime by the outgoing null hypersurfaces C_u. The acoustical quantities are the geometric quantities defined by this foliation. The acoustical structure equations are the equations satisfied by the optical quantities. The acoustical quantities of 0th order are μ and $\not{h}$, the induced metric on the $S_{t,u}$ sections of the C_u The acoustical quantity of 1st order, other than the 1st derivatives of μ, is the second fundamental form χ of the C_u, which, in accordance with the preceding discussion, is a quantity intrinsic to C_u. It is related to

$\not h$ by the first variation equation

$$\not\mathcal{L}_L \not h = 2\chi$$

one of the basic acoustical structure equations. The other acoustical structure equations depend on the ψ_α and their derivatives with respect to L and T, and the covariant derivatives with respect to $\not h$ of the restrictions of these to the $S_{t,u}$, the ψ_α being the 1st variations corresponding to the generators of translations $\partial/\partial x^\alpha$ of the underlying Minkowski spacetime.

The most important acoustical structure equation from the point of view of the formation of shocks is the propagation equation for μ along the generators of C_u:

$$L\mu = m + \mu e$$

The function m is given by:

$$m = \frac{1}{2}(\psi_L)^2 TH$$

Here H is the function of σ defined by:

$$\sigma H = 1 - \eta^2$$

Also, $\psi_L = L^\alpha \psi_\alpha$ where the $L^\alpha = Lx^\alpha$ are the rectangular components of L. It is the function m which determines shock formation, when being negative, causing μ to decrease to zero.

A fact which is crucial to the whole approach is that the derivatives of the rectangular coordinates x^α with respect to the acoustical coordinates (t, u, ϑ) can all be expressed in terms of the acoustical quantities μ and χ.

We consider the 1st variations, in particular the functions ψ_α, to be of order 0. Consider the higher order variations arising from the ψ_α. For variations ψ of order k, these are then quantities of order $k-1$. The integrants of the corresponding integrals $\mathcal{E}_0^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{E}_1^{\prime u}[\psi](t)$, $\mathcal{F}_1^{\prime t}[\psi](u)$, $K[\psi](t,u)$ are then quantities of order k as they involve $d\psi$. We define each of the five quantities

$$\mathcal{E}^u_{0,[n]}(t), \;\; \mathcal{F}^t_{0,[n]}(u), \;\; \mathcal{E}'^u_{1,[n]}(t), \;\; \mathcal{F}'^t_{1,[n]}(u), \;\; K_{[n]}(t,u) \tag{34}$$

which together give nth order control of the solution, to be the sum of the corresponding quantity $\mathcal{E}_0^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{E}_1'^u[\psi](t)$, $\mathcal{F}_1'^t[\psi](u)$, and $K[\psi](t,u)$, over all variations ψ of this type, up to order n.

The source functions $\tilde{\rho}$ which are associated to the higher order variations ψ give rise to error integrals, that is to spacetime integrals of contributions to the divergence of the corresponding energy currents. The expressions for the source functions associated to the nth order variations show that these contain the $n-1$th derivatives of the deformation tensors of the commutation fields, which in turn contain the $n-1$th derivatives of χ and nth derivatives of μ. Thus to achieve closure, we must obtain estimates for the latter in terms of the quantities (34). The analysis of the terms in the top order source functions which contain the top order spatial derivatives of the acoustical quantities shows that these terms can be expressed in terms of the 1-forms $\not{d}(R_{i_l}...R_{i_1}\mathrm{tr}\chi)$ and the functions $R_{i_{l-m}}...R_{i_1}(T)^m \not{\Delta}\mu \ : \ m=0,...,l$ where $l=n-2$. Now, it turns out that there is a function $\check{f}$ of order 1 but not containing acoustical terms of order 1 such that the order $l+2$ 1-form

$$^{(i_1...i_l)}x_l = \mu\not{d}(R_{i_l}...R_{i_1}\mathrm{tr}\chi) + \not{d}(R_{i_l}...R_{i_1}\check{f})$$

satisfies a propagation equation along the generators of the C_u the right hand side of which is again of order $l+2$. Similarly, there is a function $\check{f}'$ of order 2 but not containing acoustical terms of order 2 such that the order $l+2$ function

$$^{(i_1...i_{l-m})}x'_{m,l-m} = \mu R_{i_{l-m}}...R_{i_1}(T)^m \not{\Delta}\mu - R_{i_{l-m}}...R_{i_1}(T)^m \check{f}'$$

satisfies a propagation equation along the generators of the C_u the right hand side of which is again of order $l+2$.. It is with the help of these equations that the sought for estimates for $\not{d}(R_{i_l}...R_{i_1}\mathrm{tr}\chi)$ and $R_{i_{l-m}}...R_{i_1}(T)^m \not{\Delta}\mu$ are obtained.

The appearance of the factor of μ, which vanishes where shocks originate, in front of $\not{d}R_{i_l}...R_{i_1}\mathrm{tr}\chi$ and $R_{i_{l-m}}...R_{i_1}(T)^m \not{\Delta}\mu$ in the definitions of $^{(i_1...i_l)}x_l$ and $^{(i_1...i_{l-m})}x'_{m,l-m}$ above, makes the analysis quite delicate. This is compounded with the difficulty of the slow decay in time which the addition of the terms $-\not{d}R_{i_l}...R_{i_1}\check{f}$ and $R_{i_{l-m}}...R_{i_1}(T)^m\check{f}'$ forces. The analysis requires a precise description of the behavior of μ itself, given by certain propositions, and a separate treatment of the condensation regions, where shocks are to form, from the rarefaction regions, the terms referring not to the fluid density but rather to the density of the stacking of the C_u. The top order spatial derivatives of the acoustical quantities appearing in the top order source functions generate borderline error integrals. To overcome the difficulties the following weight function is introduced:

$$\overline{\mu}_{m,u}(t) = \min\{\mu_{m,u}(t), \eta_0\}, \quad \mu_{m,u}(t) = \min_{\Sigma_t^u} \mu$$

where Σ_t^u is the exterior of $S_{t,u}$ in Σ_t, and the quantities (34) are weighted with a power, $2a$, of this weight function.

In deriving the energy estimates of top order, $n = l + 2$, the power $2a$ of the weight $\overline{\mu}_{m,u}(t)$ is chosen suitably large to allow us to transfer the terms contributed by the borderline integrals to the left hand side of the inequalities resulting from the integral identities associated to the multiplier fields K_0 and K_1. Once the top order energy estimates are established, I revisit the lower order energy estimates using at each order the energy estimates of the next order in estimating the error integrals contributed by the highest spatial derivatives of the acoustical quantities at that order. I then establish a descent scheme, which yields, after finitely many steps, estimates for the five quantities $\mathcal{E}^u_{0,[k]}(t)$, $\mathcal{F}^t_{0,[k]}(u)$, $\mathcal{E}'^u_{1,[k]}(t)$, $\mathcal{F}'^t_{1,[k]}(u)$, and $K_{[k]}(t,u)$, for $k = l + 1 - [a]$, where $[a]$ is the integral part of a, in which weights no longer appear.

It is these unweighted estimates which are used to close the bootstrap argument by recovering the bootstrap assumptions that have been used and which reduce to pointwise estimates for the variations up to certain order. This is accomplished by the method of continuity through the use of the isoperimetric inequality on the $S_{t,u}$, and leads to the main theorem.

After the proof of the main theorem [C2] turns to the study of the structure of the boundary of the domain of the maximal development and the behavior of the solution at this boundary. The boundary consists of a regular part $\underline{C}$ and a singular part B. Each component of $\underline{C}$ is a regular incoming acoustically null hypersurface with a singular past boundary which coincides with the past boundary of an associated component of B. The union of these singular past boundaries we denote by $\partial_- B$. Each component of B is a hypersurface which is smooth relative to both differential structures and has the intrinsic geometry of a regular null hypersurface in a regular spacetime and, like the latter, is ruled by invariant curves of vanishing arc length. On the other hand, the extrinsic geometry of each component of B is that of an acoustically spacelike hypersurface which becomes acoustically null at its past boundary, an associated component of $\partial_- B$. This means that at each point $q \in B$ the past null geodesic conoid of q does not intersect B. Each component of $\partial_- B$ is an acoustically spacelike surface which is smooth relative to both differential structures. A result of the last part of [C2] is the *trichotomy theorem*. According to this theorem, for each point q of $\partial_- B \bigcup B$ the intersection of the past null geodesic conoid of q with any Σ_t in the past of q splits into three parts, the parts corresponding to the outgoing and to the incoming sets of null geodesics ending at q being embedded discs with a common boundary, an embedded circle, which corresponds to the set of the remaining null geodesics ending at q. *All outgoing null geodesics ending at q have the same tangent vector at q*. This vector is then an invariant null vector associated to the singular point q, the tangent vector to the invariant null curve through q if $q \in B$, initiating at q if $q \in \partial_- B$. This is in fact the reason why the considerable freedom in the choice of the acoustical function does not matter in the end. For, considering the transformation from one acoustical function to another, I show that the foliations corresponding to different families of outgoing null hypersurfaces have equivalent geometric properties and degenerate in precisely the same way on the same singular boundary.

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Calls for nominations

Call for nominations for the Dubrovin Medal

The Boris Dubrovin medal, in memory of Boris Anatolievich Dubrovin, Professor at the International School for Advanced Studies (SISSA) in Trieste from 1993 to 2019, is awarded by SISSA, with the support of the Moscow Mathematical Society, the Gruppo Nazionale per la Fisica Matematica and the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni, which are part of the Istituto Nazionale di Alta Matematica. Boris Dubrovin made several groundbreaking contributions to mathematical physics, with far-reaching influences in various areas of mathematics, see e.g. the obituary. Possibly one of his best-known inventions is the definition of Frobenius manifolds (which we ought now to refer to as "Dubrovin-Frobenius manifolds"). Their formalization built a surprising bridge between mathematical physics and differential, algebraic and symplectic geometry. The medal recognizes exceptionally promising young researchers who have already made outstanding contributions to the fields of mathematical physics and geometry. The Dubrovin medal is awarded every two years, starting from 2020.

The statutes of the Boris Dubrovin medal can be found here.

The call for the 2022 nomination is available here. The deadline for nominations is February 28, 2022.

The 2022 Dubrovin medal is sponsored by Letters in Mathematical Physics and SISSA Medialab.

Call for nominations, Eisenbud Prize

The Leonard Eisenbud Prize for Mathematics and Physics is awarded every three years by the American Mathematical Society to honor "a work or group of works, published in the preceding six years, that brings mathematics and physics closer together" with a cash award of $5000. The nomination period is open and lasts thropugh June, 2022. Further details and the procedure to make nominations are available at https://www.ams.org/prizes-awards/paview.cgi?parent_id=23

Time's Arrow

Scientific anniversaries

1822.
In March, Navier read his "Mémoire sur les lois du mouvement des fluides" at the Royale Académie des Sciences.

Fourier's *Théorie analytique de la chaleur* appeared.

1922.
In January, Bohr introduced the notion of atomic shells in Fysisk Tidsskrift (later in the year reprinted in German translation in Zeitschrift für Physik).

Also in January, Stern and Gerlach demonstrated the quantization of angular momentum, as published in December in Zeitschrift für Physik.

1997.
In March this *News Bulletin* began electronic publication.

Lost luminaries

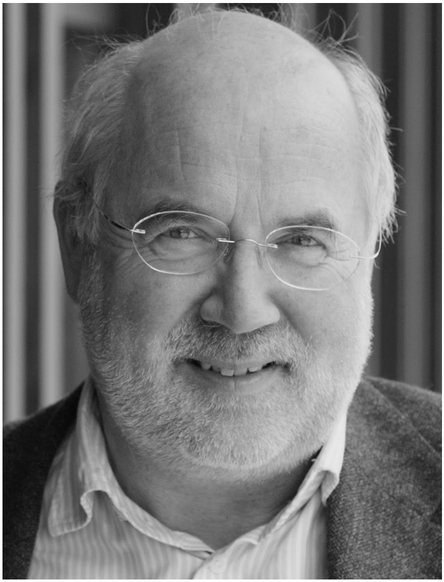
Krzysztof Gawędzki, 21 January, 2022.

Robert Strichartz, 19 December, 2021.

This issue contains an obituary for Detlef Dürr, whose demise was previously announced in "Time's Arrow."

Readers are encouraged to send items for "Time's Arrow" to bulletin@iamp.org.

Detlef Dürr (1951–2021)



We mourn the unexpected and untimely passing of Professor Detlef Dürr on January 3, 2021, at age 69 after brief and severe illness. Until few weeks before, he was active in research and teaching at Ludwig-Maximilians-Universität in Munich, where he had served as a professor of mathematics since 1989. He will be dearly missed by his friends, his students, his colleagues, and an international scientific community that he has influenced significantly.

Detlef Dürr was an unusual scientist and an unusual man in many ways. He was particularly known as a passionate advocate of Bohmian mechanics, which he regarded as a reasonable theory of quantum physics, and as a critic of the Copenhagen interpretation, which he rejected as scientifically unsatisfactory. He coined the expression "Bohmian mechanics" in analogy to and distinction from Newtonian mechanics. With his long-term collaborators and friends Sheldon Goldstein (Rutgers University, USA) and Nino Zanghì (University of Genoa, Italy), he made seminal contributions from 1992 onwards to our mathematical and physical understanding of this theory and of quantum physics in general. Dürr, Goldstein, and Zanghì also showed how the paradoxes of the traditional view of quantum physics are resolved in Bohmian mechanics in a natural way. At the same time, Detlef Dürr was not at all dogmatic as an advocate of Bohmian mechanics; he actually emphasized that theories such as the one of Ghirardi-Rimini-Weber, which propose a stochastic modification of the Schrödinger equation in order to account for the collapse of the wave function, can provide an alternative possibility for how the quantum world might work. He authored several research works himself about such collapse theories, and was friends with their leading advocates GianCarlo Ghirardi and Angelo Bassi.

Detlef Dürr was born on March 4, 1951, in Hänigsen (Germany). He attended the University of Münster and was awarded a Ph.D. there in 1978 with a thesis on the most probable paths of diffusion processes. Afterwards, he did postdoctoral work at Rutgers University with Joel Lebowitz and at Ruhr-University Bochum and Bielefeld University with Sergio Albeverio about mathematical and conceptual questions of statistical mechanics. In Bochum he received the habilitation in 1983 with a thesis on a derivation of diffusion processes as trajectories of a labeled particle within the deterministic motion of a system of billiard balls in the limit of many balls. In statistical mechanics, it was important to him to resolve common misunderstandings of the subject. He supported Boltzmann's explanation of the second law of thermodynamics and emphasized the relevance of typicality for the origin of statistical laws in deterministic theories such as Bohmian or Newtonian mechanics.

During his time in Munich, he maintained particularly close scientific and personal contact with Herbert Spohn, with whom he held a weekly seminar. With Goldstein and Zanghi, Detlef was able to show in a seminal paper in 1992 that in typical universes governed by Bohmian mechanics, the outcomes of experiments obey the quantum-mechanical measurement formalism. Thus, the axioms of quantum mechanics are theorems in Bohmian mechanics. Since then, his research focused particularly on the foundations of quantum mechanics. However, he also conducted research on classical electrodynamics according to the approach of Wheeler and Feynman, on electron-positron pair creation, the theory of the Dirac sea, and shape space dynamics, among other topics. His scientific work was grounded in the view that foundational questions in physics can and must be treated with the same high standards as other fields of physics.

It was particularly dear to Detlef to reach the younger generations, to nourish their fascination with science and to let them participate early on in the scientific community. He has advised an unusually large number of doctoral and other theses. He attracted many students through his enthusiastic teaching style and his focus on fundamental questions. In his working group at Ludwig-Maximilians-Universität, he created an exceptional culture of scientific discussion. Beside his research and teaching activities and particularly during his tenure as the Director of the Mathematics Institute, he was dedicated to improve the education of future mathematics teachers. He has written three textbooks (on Bohmian mechanics, quantum mechanics, and probability theory) that each differ substantially from ordinary books on similar subjects. They each pay particular attention to foundational questions that usually get neglected.

It was Detlef's view that a full understanding of physics includes a good grasp of mathematics and philosophy. Accordingly, he sought and supported the interdisciplinary exchange. For example, a series of conferences at the Bielefeld Center for Interdisciplinary Research that he initiated and organized under the title "Quantum Theory Without Observers" (1995, 2004, 2013) brought together leading researchers from these disciplines. Over the years, he organized numerous summer schools and workshops, and for this purpose he founded, together with New York philosopher Tim Maudlin, the "John Bell Institute for the Foundations of Physics" a few years ago.

Although he enjoyed controversial scientific discourse and was not afraid of provocative statements, he was always respectful and strove for mutual understanding. With his cordiality he was able to build many bridges between different camps. This he also did by singing and playing the guitar. He had found his love for music in his youth together with his brother, and in 1974 he even published a record jointly with fellow guitarist Peter Finger. At some conferences, he gave not only a scientific talk but also a concert.

In Detlef Dürr we have lost a visionary scientist, an engaging teacher, and a good friend. With him departs a piece of scientific and humane culture. We will miss him dearly.

Gernot Bauer (Münster)
Dirk-André Deckert (Munich)
Peter Pickl, Stefan Teufel, Roderich Tumulka (Tübingen)

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new member:

Dr. Julio Hoff da Silva, Brazil

Conference announcements

From Operator Theory to Orthogonal Polynomials, Combinatorics, and Number Theory

May 23 - 27, 2022, Baylor University, Waco, TX, USA

Mathematical physics conferences related to the International Congress of Mathematicians

- 2022, the year of the St. Petersburg ICM, will mark the centenary of outstanding mathematician Olga Ladyzhenskaya who was a one of the world leaders in the field of Partial Differential Equations and Mathematical Hydrodynamics in the second half of the twentieth century. Two scientific events in St. Petersburg are dedicated to this anniversary.

  The conference ***Mathematical hydrodynamics: the legacy of Olga Ladyzhenskaya and modern perspectives*** will be held during May 23-27 in Leonhard Euler International Mathematical Institute.

  https://indico.eimi.ru/event/641/

  The second event, ***O.A. Ladyzhenskaya centennial conference on PDEs*** will be held on July 16-23, just after ICM, in St. Petersburg Department of Steklov Mathematical Institute (PDMI). This conference is an ICM 2022 satellite.

  http://www.pdmi.ras.ru/EIMI/2022/Ladyzhenskaya/index.html

- Spectral theory and mathematical physics have been among the highlights of St. Petersburg mathematical school over many decades. The names of Ludwig Faddeev, Mikhail Birman, and Vladimir Buslaev are among the names of worldwide leaders in these fields. Several generations of researchers from St. Petersburg, working both in Russia and abroad, have been successful in defining a significant part of the international mathematical physics and spectral theory community.

  The ICM 2022 Satellite ***St. Petersburg Conference in Spectral Theory and Mathematical Physics, dedicated to the memory of M.Sh. Birman*** will be held on June 22-26, in Leonhard Euler International Mathematical Institute.

  https://indico.eimi.ru/event/281/

- The conference ***Integrability and Moduli, dedicated to Leon Takhtajan's 70th birthday***, will be held during October 24-28, 2022, in Leonhard Euler International Mathematical Institute.

  https://sites.google.com/view/integrability-and-moduli/home

  This conference, postponed due to the pandemie, is aimed to celebrate the 70th birthday of Leon Takhtajan. His outstanding contributions to mathematical physics and Teichmuller theory are well known, and his pioneering works on the theory of classical and quantum integrable systems are widely acknowledged. In particular, influenced by ideas of Conformal Field Theory, he studied the Liouville action as a pure mathematical object and related it to the theory of Riemann surfaces and vector bundles. World top experts in these fields plan to attend the conference.

Open positions

For an updated list of academic job announcements in mathematical physics and related fields visit

`http://www.iamp.org/page.php?page=pagepositions`

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