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Contents

Changes at the *Bulletin*

The *Bulletin* is pleased to welcome Ian Jauslin to the Editorial Board and to announce the appointment of Rafael Benguria as Editor in Chief, beginning with the January issue. Evans Harrell will continue on the Editorial Board.

Hydrodynamic Limits: the emergence of fractional boundary conditions

by Patrícia Gonçalves

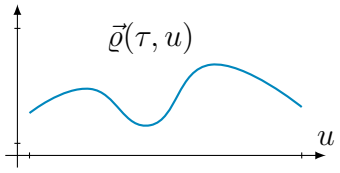
In these notes I describe some recent developments around the hydrodynamic limit for some stochastic interacting particle systems that have been investigated by a group of researchers working under the research project funded by the ERC Starting grant no. 715734. The focus will be given on stochastic systems with an open boundary, for which one can obtain partial differential equations with boundary conditions; or stochastic systems with long range interactions, for which fractional equations appear in the scaling limits of those models. For a matter of size and taste, this is, by no means, an extensive review about the subject, the chosen topics have the flavour from a personal perspective of the author.

1 Introduction

Deriving rigorously the evolution equations of classical fluid mechanics from the large-scale description of the conserved quantities in Newtonian particle systems is a long-standing problem in mathematical physics. More precisely, the area of statistical mechanics dedicates to understanding the emergence of evolution laws from the kinetic description of the underlying system of particles. To attack this problem, we can assume that the motion of particles is random. We introduce two scales: a macroscopic scale, where the systems' thermodynamical quantities are analysed, as e.g. density, pressure, temperature, etc, that we denote by $\vec{\varrho} := (\varrho_1, \cdots, \varrho_n)$. The other scale, the microscopic scale, is the scale at which the particles of the system are analysed as a whole. As a possible scenario, one can be interested in understanding the physical evolution of a gas confined to a finite volume. The number of molecules is of the order of Avogadro's number, therefore, one cannot give a precise description of the microscopic state of the system, but the goal consists in describing the macroscopic behaviour from the random movement of the molecules.

Understanding the connection between macro/micro spaces is one of the goals in statistical mechanics and according to one of the pillars of this area, Ludwig Boltzmann, first, we should determine the stationary states of the system under investigation, let us denote them by μ and then we should characterize them in terms of the thermodynamical quantities of interest $\vec{\varrho}$, obtaining then $\mu_{\vec{\varrho}}$. Finally we can analyse the evolution of the system out of equilibrium. To formalize this problem from the mathematical point of view, consider a macroscopic space Λ and fix an arbitrary point u and a small neighbourhood $\mathcal{V}_u$ around it, in such a way that it is macroscopically small but big enough to contain infinitely many molecules. Due to their strong interaction, we can assume the system to be locally in equilibrium, so that the state of the system at the point u should be close to $\mu_{\vec{\varrho}(u)}$. Observe that this local equilibrium is characterized by the thermodynamical

quantities $\vec{\varrho}$ that now depend on the position u. We let time evolve, and we assume that the local equilibrium persists at a longer time. Later on, we stop the system at some time τ and now the local equilibrium will be given in terms of $\vec{\varrho}(\tau, u)$, depending both on time and space, i.e. the state of the system should be close to $\mu_{\vec{\varrho}(t,u)}$. The function $\varrho(\vec{t}, u)$ should then evolve according to some PDE, the so-called *hydrodynamic equation*.



As mentioned above, treating this problem from the mathematical point of view is challenging and some simplifications are usually assumed. A possible approach is to consider that the dynamics of particles is random, from where it arises the commonly known *stochastic interacting particle systems* (SIPS), which are random systems typically used in statistical mechanics to attack this sort of problems. Back in the seventies, these systems were introduced in the mathematics community by Spitzer in [50], but were already known to physicists and biophysics since the seminal article of MacDonald, Gibbs and Pipkin [47]. The dynamics of these systems conserve a certain number of quantities. At the micro level one assumes that each molecule behaves as a continuous-time random walk evolving in a proper discretization of the macroscopic space Λ and this allows for a probabilistic analysis of the discrete system. For details on the formal definition of SIPS, we refer to the seminal book of Liggett [46]. The obtained molecules' evolution is Markovian, so their future evolution conditioned to their past, only depends on the knowledge of the present. We can discretize the volume Λ according to a scaling parameter $\epsilon > 0$. At each site of the discrete set we can put randomly a certain number of particles, and repeat this independently on all the other sites. Thus we have just fixed the initial state of the system. Each one of these particles waits an exponentially distributed time after which one of them jumps to some other site, if the dynamical rules allow for it. Once the dynamics is fixed, according to Boltzmann, one should find the stationary measures, and characterize them in terms of the relevant thermodynamical quantities.

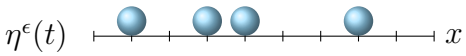
The goal, in the hydrodynamic limit, consists in obtaining the PDEs, that govern the space-time evolution of each conserved quantity of the system [45, 51]. The macroscopic and microscopic spaces will be connected by means of the scaling parameter ϵ so that the typical distance between particles is of order ϵ. At the end ϵ will be taken to 0. To observe a non-trivial impact at the macroscopic level from the particles' motion, one has to look at the system in a longer time scale $\tau(\epsilon)$, that depends on the scaling parameter ϵ and on the dynamical rules. If the dynamical rules allow a strong long range interaction then the time needed for a macroscopic effect is shorter with respect to dynamics that allow very short range interactions.

2 Hydrodynamic Limit

In order to exhibit some PDEs that can be obtained for some SIPS, in the next subsections we describe the hydrodynamic limit for a system with only one conservation law and then we discuss the case with more conservation laws.

2.1 A classical SIPS: the exclusion process

The model: One of the most classical SIPS is the exclusion process whose dynamics can be described as follows. Recall that ϵ is the scaling parameter connecting the macroscopic space Λ and the microscopic space Λ_ϵ. Assume that at each site of Λ_ϵ there can be at most one particle (the so called exclusion rule), so that if η is a configuration, then $\eta_x(t)$ denotes the number of particles at site x and at time t, and $\eta_x(t) \in \{0,1\}$. To each bond $\{x,y\}$ of Λ_ϵ it is attached a Poisson process of rate one. The trajectories of Poisson processes are discontinuous and at each site where the discontinuity occurs we say that there is a mark of the Poisson process. Poisson processes attached to different bonds are independent. This means that particles have to wait for a random time which is exponentially distributed with mean one, and when there is a mark of the Poisson process associated to a bond $\{x',y'\}$, the particles at that bond exchange positions at rate $p(y'-x')$, where $p:\mathbb{Z}\to[0,1]$ is a transition probability. The jump happens if, and only if, the exclusion rule is obeyed, otherwise the particles wait for another mark of one Poisson process. The number of particles in the system is fixed by its initial state and since this dynamics only exchanges particles along the microscopic space, the density is a conserved quantity.



The state-space is $\{0,1\}^{\Lambda_\epsilon}$ and when jumps are allowed only to nearest-neighbors, the process is said to be simple. First, we explain the phenomenology in case of nearest-neighbor jumps, and then, the extension to the long jumps case. To that end, for now, we consider $p(-1) = 1-p(1)$ and $p(1) = p + E\epsilon^\kappa$ where $p\in[0,1]$ and $E,\kappa \geq 0$. If $E=0$ and $p=1/2$, we obtain the extensively studied symmetric simple exclusion process (SSEP); while if $E=0$ but $p\neq 1/2$ then we get the asymmetric simple exclusion process (ASEP) and if $E\neq 0$ and $p=1/2$, we get the weakly asymmetric simple exclusion process (WASEP). Observe that the parameter κ rules the strength of the asymmetry. The infinitesimal generator of this process is given on $f:\{0,1\}^{\Lambda_\epsilon}\to\mathbb{R}$ and $\eta\in\{0,1\}^{\Lambda_\epsilon}$ by

$$\mathcal{L}^{ex} f(\eta) = \sum_{x\in\Lambda_\epsilon} \Big\{ p(1)\eta_x(1-\eta_{x+\epsilon}) + p(-1)\eta_{x+\epsilon}(1-\eta_x) \Big\} \nabla_{x,x+\epsilon} f(\eta),$$

where $\nabla_{x,x+\epsilon} f(\eta) = f(\eta^{x,x+\epsilon}) - f(\eta))$, $\eta^{x,x+\epsilon}$ is the configuration obtained from η by swapping the occupation variables at x and $x+\epsilon$. We can think of this operator as a differential operator that when testing functions defined of the state space of the process, it gives a weight which is the product between the jump rate and the difference between

the value of the function f evaluated at the configurations after and before the jump. This operator corresponds to the time derivative of the semigroup S_t of the process defined through the formula

$$\mathcal{L}^{ex} f(\eta) := \lim_{t\to 0} \frac{S_t f(\eta) - f(\eta)}{t}.$$

Now we speed the system in the time scale $t\tau(\epsilon) = t\epsilon^{-a}$, where $a > 0$ will be chosen ahead in order to see a non-trivial macroscopic evolution. The system conserves one quantity: the number of particles $\sum_{x\in\Lambda_\epsilon} \eta_x$. Now, we should obtain the stationary measures of this process and parametrize them by a constant density ϱ. By this we mean that if we denote by ν_ϱ a stationary measure of the process, then if the initial process has distribution ν_ϱ, i.e. the law of η_0 is given by ν_ϱ, then at any time t, the same holds i.e. the law of η_t is given again by ν_ϱ. For the exclusion processes defined above, the space-time invariant measures are Bernoulli product measures of parameter $\varrho \in [0,1]$:

$$\nu_\varrho(d\eta) = \prod_{x\in\Lambda_\epsilon} \varrho^{\eta_x}(1-\varrho)^{1-\eta_x}, \tag{1}$$

and in fact, these measures are reversible so some choices of $p(\cdot)$. And this means that the adjoint generator $(\mathcal{L}^{ex})^*$ in the Hilbert space ${}^2(\nu_\varrho)$, coincides with the generator $\mathcal{L}^{ex}$.

Hydrodynamic limit of exclusion processes: The empirical measure associated to the number of particles is given on $\eta \in \{0,1\}^{\Lambda_\epsilon}$ by

$$\pi^\epsilon(\eta, du) := \epsilon \sum_{x\in\Lambda_\epsilon} \eta_x \, \delta_x \, (du) \,, \tag{2}$$

where δ_x is a Dirac mass at x. Observe that for a given configuration η the measure $\pi^\epsilon(\eta, du)$ gives weight ϵ to each particle. We define the process of empirical measures as $\pi^\epsilon_t(\eta, du) = \pi^\epsilon(\eta(t\tau(\epsilon)), du)$.

The rigorous statement of the hydrodynamic limit says that, given a measurable profile $\varrho(0,u)$, if the process starts from a probability measure μ_ϵ for which a Law of Large Numbers (LLN) for $\pi^\epsilon_0(du)$ holds, i.e.

$$\pi^\epsilon_0 \to_{\epsilon\to 0} \varrho(0,u)du,$$

then the same result holds at any time t, i.e.

$$\pi^\epsilon_t \to_{\epsilon\to 0} \varrho(t,u)du,$$

where $\varrho(t,u)$ is the solution (in some sense) of the hydrodynamic equation. Observe that the assumption above says that the random measure $\pi^\epsilon_0(du)$ converges, as $\epsilon \to 0$, in the weak sense, to the deterministic measure $\varrho(0,u)du$. This means that for any given continuous function f it holds

$$\lim_{\epsilon\to 0} \Big| \int_\Lambda f(u)\pi^\epsilon_0(\eta, du) - \int_\Lambda f(u)\varrho(0,u)du \Big| = 0.$$

But we still need to say in which sense the convergence holds, because the left-hand side of last display is still random. We therefore assume that the previous result holds in probability with respect to μ_ϵ, i.e. for any $\delta > 0$, it holds

$$\lim_{\epsilon\to 0} \mu_\epsilon\Big(\eta : \Big| \int_\Lambda f(u)\pi_0^\epsilon(\eta, du) - \int_\Lambda f(u)\varrho(0,u)du\Big| > \delta\Big) = 0.$$

And this will be a restriction on the set of initial measures for which the result will be derived.

Hydrodynamic equations: To give an intuition of which equations can be derived from SIPS we give now an heuristic argument for the exclusion processes defined above. Recall that for these processes, the invariant measures are the Bernoulli product with marginals given in (1). Consider the discrete profile given by

$$\rho_t^n(x) = \mathbb{E}[\eta_t(x)].$$

From Kolmogorov's equation we have that

$$\partial_t \rho_t^n(x) = \mathbb{E}[\mathcal{L}^{ex}\eta_x(t)]$$

and a simple computation shows that

$$\mathcal{L}^{ex}\eta(x) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$$

where $j_{x,x+1}(\eta)$ denotes the instantaneous current at the bond $\{x, x+1\}$. Let us know assume that the process is the SSEP. In this case

$$j_{x,x+1}(\eta) = \eta_x(1-\eta_{x+1}) - \eta_{x+1}(1-\eta_x) = \eta_x - \eta_{x+1}.$$

Since $j_{x,x+1}$ is the gradient of η_x, we get

$$\partial_t \rho_t^n(x) = \mathbb{E}[\Delta_n \eta_x],$$

where Δ_n denotes the discrete Laplacian. Above the expectation $\mathbb{E}$ is with respect to the Bernoulli product measure given in (1), but with a parameter given by $\rho_t^n(\cdot)$. Now, if we assume that for all x it holds $\lim_{n\to\infty} \rho_t^n(x) = \rho_t(\frac{x}{n})$, then the evolution of the density is given by the heat equation:

$$\partial_t \rho_t(u) = \Delta \rho_t(u).$$

Of course that we are assuming a local equilibrium property above, but this heuristic argument can be made rigorous by some methods and for many different models.

For the exclusion process introduced above, we can get as hydrodynamic equations [20, 43, 45]:

a. SSEP with $a = 2$, the heat equation

$$\partial_t \varrho = \tfrac{1}{2}\Delta\varrho; \tag{3}$$

b. WASEP with $\kappa = 1, a = 2$, the viscous Burgers equation

$$\partial_t \varrho = \tfrac{1}{2}\Delta \varrho + E\nabla F(\varrho); \tag{4}$$

c. ASEP and $a = 1$, the inviscid Burgers equation

$$\partial_t \varrho = E\nabla \varrho(1-\varrho). \tag{5}$$

For symmetric $p(\cdot)$, i.e. $p(z) = p(-z)$ for all $z \in \mathbb{Z}$, allowing long jumps with infinite variance, e.g.

$$p(z) = c_\gamma/|z|^{1+\gamma}\mathbf{1}_{z\neq 0}. \tag{6}$$

we obtain the fractional heat equation i.e. $\partial_t \varrho = [-(-\Delta^{\gamma/2})\varrho]$ for $\gamma \in (0,2)$, see [42]. Note that the infinite variance case corresponds to $\gamma \in (0,2)$ since in that case $\sum_z z^2 p(z) = \infty$. In case $p(\cdot)$ is asymmetric one can obtain an integro-PDE [49]. All these equations can be supplemented with several boundary conditions by superposing the dynamics described above with another one as, for example:

1. Considering the exclusion process evolving on the lattice

$$\Lambda_\epsilon = \{0, \epsilon, 2\epsilon, \ldots, \epsilon^{-1}\epsilon = 1\}$$

and adding at the boundary points $x = 0$ and $x = 1$, a dynamics that injects particles (at rate $\alpha\epsilon^\theta/\beta\epsilon^\theta$ at the left/right reservoir) or removes particles (at rate $(1-\alpha)\epsilon^\theta/(1-\beta)\epsilon^\theta$ at the left/right reservoir) in the system.

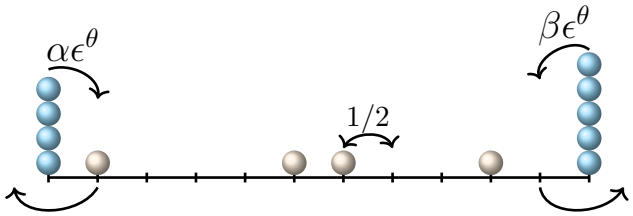


Figure 2.1: Symmetric exclusion with open boundary.

The parameters satisfy $\alpha, \beta \in [0,1]$ and $\theta \in \mathbb{R}$. Note that the conservation law is destroyed in this case, but inside the system, it is still conserved.

2. Considering the exclusion process with a dynamics that block the passage of particles between certain regions of Λ_ϵ (the conservation law is maintained in this case). For example, assume that the exchange rate of particles in a certain number of bonds is given by a transition probability $p(\cdot)$, while in some other bonds this rate is multiplied by a factor that makes it slower with respect to the rate in all other bonds. In the next figure, particles jump everywhere in $\Lambda_\epsilon = \epsilon\mathbb{Z}$, but the jump rate for bonds in $\epsilon\mathbb{Z}_+$ or $\epsilon\mathbb{Z}_-$ is given by $p(\cdot)$, but the jump rate between sites in $\epsilon\mathbb{Z}_+$ and $\epsilon\mathbb{Z}_-$ is given by $p(\cdot)\alpha\epsilon^\beta$, where now the parameters satisfy $\alpha > 0$ and $\beta \geq 0$.

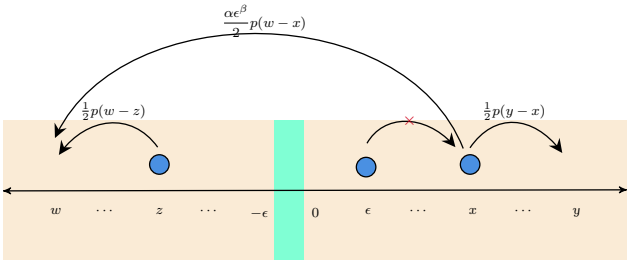


Figure 2.2: Long-jumps symmetric exclusion with a slow barrier.

Under this choice we are creating a slow barrier at the macroscopic level and the goal is to understand how these local microscopic defects propagate to the macroscopic level. In this case we do not have a superposition of two dynamics as in the previous case, we are solely slowing down the dynamics in certain places of the microscopic space.

For recent results on 1. we refer to [3, 19, 26, 22, 23] for the SSEP in contact with slow/fast boundary reservoirs. In that case, the heat equation has boundary conditions with Dirichlet, Robin, or Neumann type, depending on the intensity of the reservoirs' dynamics. More precisely, we can get the heat equation given in (3) with the boundary conditions:

(I) Dirichlet : $\varrho_t(0) = \alpha$, $\varrho_t(1) = \beta$, if $\theta < 1$.

(II) Robin : $\partial_u \varrho_t(0) = \varrho_t(0) - \alpha$, $\partial_u \varrho_t(1) = \beta - \varrho_t(1)$, if $\theta = 1$.

(III) Neumann : $\partial_u \varrho_t(0) = \partial_u \varrho_t(1) = 0$, if $\theta > 1$.

For the WASEP one can get the viscous Burgers equation as in (4) with Dirichlet conditions as in (I) or with Robin boundary conditions, but in this case the boundary conditions are non-linear, see [14]. For the ASEP, the parabolic equations obtained above are replaced by hyperbolic laws and with several boundary conditions [2, 54].

For the dynamics defined in 1., but in case of long jumps, we refer to [9, 10, 4] where it is considered the transition probability given in (6), superposed with a dynamics that injects and removes particles in the system and that acts everywhere in Λ_ϵ with a strength regulated again by a parameter $\theta \in \mathbb{R}$, see Figure 2.3.

Depending on the finiteness of the variance of the transition probability $p(\cdot)$ and the strength of the Glaubler dynamics, the scheme with the results for the hydrodynamic limit is extremely rich, since one can obtain different operators appearing at the macro level and the equations have several boundary conditions of fractional form.

When the transition probability $p(\cdot)$ has finite variance i.e. $\sum_z z^2 p(z) < +\infty$, which holds for $\gamma > 2$, the hydrodynamic equation for $a = 2$ is the heat equation with several boundary conditions. When $\gamma = 2$, the variance diverges as $\log(\epsilon)$ and to compensate this diverging term, we have to take the time scale $\epsilon^{-2}/\log(\epsilon)$ to obtain again the heat

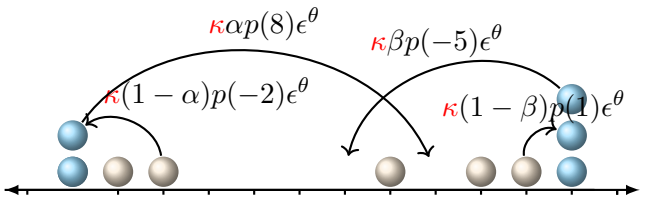


Figure 2.3: Long-jumps symmetric exclusion with a slow boundary.

equation with several boundary conditions.

When $\gamma \in (0,2)$ the variance is infinite and the system becomes super-diffusive. Therefore, the equation is no longer written in terms of the Laplacian operator but the fractional Laplacian. Since the solutions of the equation are defined on the interval $[0,1]$, one has, in fact, the regional fractional Laplacian. Now the boundary conditions involve fractional derivatives. For a summary on the regimes where the boundary conditions appear see the figure below.

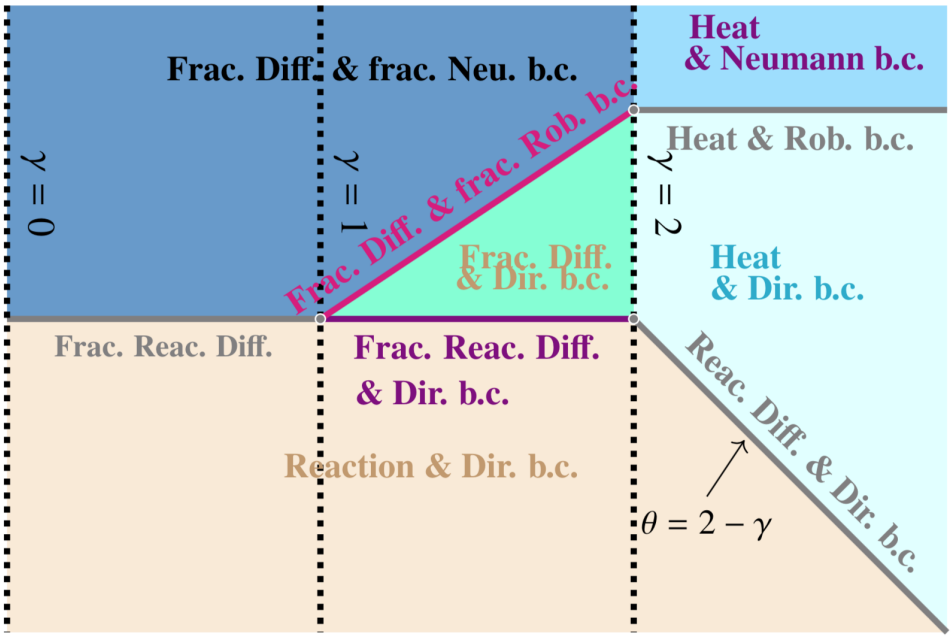


Figure 2.4: Scheme for hydrodynamic limits.

For recent results on 2. we refer the reader to [27, 28] for the SSEP with a slow bond on the torus

$$\mathbb{T}_\epsilon = \{0, \epsilon, 2\epsilon, \cdots, \epsilon^{-1}\},$$

[29] for the SSEP with a slow site on $\mathbb{T}_\epsilon$, [15, 16] for the SSEP on $\epsilon\mathbb{Z}$ with a slow barrier blocking the passage of particles.

We note that in the case of a slow barrier the scheme of hydrodynamic limits is also very rich. When the intensity of the barrier is equal to $\alpha\epsilon^\beta$ and slows down the passage of particles between negative and positive sites on $\epsilon\mathbb{Z}$, an interesting behavior appears (contrarily to the slow bond case of [27]) when $\beta = 0$:

1. for $\alpha = 1$, in [42] it is obtained the fractional heat equation.

2. for $\alpha \neq 1$, the fractional Laplacian is replaced by a regional fractional Laplacian, but in this case defined on an unbounded domain. In this case, since there are infinitely many slow bonds at the microscopic level, the impact of their slowed dynamics (which differs from the dynamics of other bonds just by a constant) is felt at the macroscopic level.

3 for $\alpha > 0$ and $\beta = \gamma - 1$, one can get fractional linear Robin boundary conditions.

4. for $\alpha > 0$ and $\beta > \gamma - 1$, one can get fractional Neumann boundary conditions.

Above we have obtained the heat equation or the fractional heat equation but we note that it is possible to obtain a non-linear version of those equations in the form:

$$\partial_t \varrho = \mathcal{P}\varrho^m;$$

where $m \in \mathbb{N}$ and $\mathcal{P} = \Delta$ or $\mathcal{P} = -(-\Delta)^{\gamma/2}$, i.e. the porous medium equation and its fractional version. For details we refer to [21, 13, 17]. To achieve these PDEs we simply can consider an exclusion dynamics where the jump rate depends on the number of particles in the vicinity of the point where particles exchange positions, see [38, 13, 17].

2.2 Two conservation laws

In this subsection we review the hydrodynamic limit for two different models with more than one conservation law. The analysis of the asymptotic behavior of the relevant quantities is much more intricate than for the case of models with just one conserved quantity, as the exclusion process described above.

The ABC model

The model: The ABC model consists of a system of particles of three species $\alpha \in \{A, B, C\}$, with exchanges only to neighbouring sites on the torus $\mathbb{T}_\epsilon$ and in the presence of a driving force, so the interaction rate depends on the type of particles involved. As in the exclusion process previously explained, at each site there is at most one particle. The total number of particles of each species is conserved. This is a continuous-time Markov process with state-space $\tilde{\Omega}_\epsilon = \{A, B, C\}^{\mathbb{T}_\epsilon}$. To properly define its hydrodynamic limit, we introduced the occupation numbers of the species α as $\xi^\alpha : \tilde{\Omega}_\epsilon \to \{0, 1\}^{\mathbb{T}_\epsilon}$ acting on the configurations in the following way $\xi^\alpha_x(\eta) = \mathbf{1}_{\{\alpha\}}(\eta_x)$. Its infinitesimal generator acts on functions $f : \tilde{\Omega}_\epsilon \to \mathbb{R}$ as

$$\tilde{\mathcal{L}}_\epsilon f(\eta) = \sum_{x \in \mathbb{T}_\epsilon} c_x(\eta)[f(\eta^{x,x+\epsilon}) - f(\eta)],$$

where the rates are defined in the following way:

$$c_x(\eta) = \sum_{\alpha,\beta} c_x^{\alpha\beta} \xi_x^\alpha \xi_{x+1}^{\alpha+1},$$

where a configuration (α, β) on the bond $\{x, x+\epsilon\}$ is exchanged to (β, α) at rate

$$c_x^{\alpha\beta} = 1 + \frac{\epsilon^\gamma (E_\alpha - E_\beta)}{2},$$

for $\alpha, \beta \in \{A, B, C\}$, where $E_\alpha \geq 0$. The role of γ in this model is to tune the strength of the driving force. This model generalizes the one introduced in [24, 25]. The system will be sped up in the time scale $a = 2$. We can think of this model as a two-species particle system, of species A and B, since the type C can be easily recovered from A and B.

We introduce the empirical measure (defined similarly to (2)) for each one of the conserved quantities ξ_x^A, ξ_x^B and ξ_x^C, i.e. for each $\alpha \in \{A, B, C\}$ we define

$$\pi^{\epsilon,\alpha}(\eta_t, du) = \epsilon \sum_{x \in \mathbb{T}_\epsilon} \xi_x^\alpha(\eta_t)\, \delta_x(du)\,. \tag{7}$$

Hydrodynamic limit of ABC: In the diffusive time scaling $a = 2$ and for $\gamma = 1$, for any t, the empirical measure

$$(\pi^{\epsilon,A}(\eta_t, du), \pi^{\epsilon,B}(\eta_t, du))$$

converges as $\epsilon \to 0$ to the deterministic measure

$$(\pi_t^A(du), \pi_t^B(du)) = (\varrho_t^A(u)du, \varrho_t^B(u)du),$$

where the densities $(\varrho_t^A(u), \varrho_t^B(u))$ solve the system of (parabolic) equations [12] given by

$$\begin{cases} \partial_t \varrho^A = \Delta \varrho^A - \nabla[F(\varrho^A)(E_A - E_C) - \varrho^A \varrho^B (E_B - E_C)] \\ \partial_t \varrho^B = \Delta \varrho^B - \nabla[F(\varrho^B)(E_B - E_C) - \varrho^A \rho^B (E_A - E_C)] \end{cases} \tag{8}$$

where $F(\varrho) = \varrho(1-\varrho)$ and for $\alpha \in \{A, B, C\}$, ϱ^α denotes the density of particles of type α in the system. The equation for the species C can easily be obtained by using the identity $\varrho^C = 1 - \varrho^A - \varrho^B$. In this case, the hydrodynamic limit is given by a system of coupled equations, since the evolution of particles of a certain species is affected by the particles of other species. One can also consider this model in contact with slow/fast reservoirs, extending the model defined above. Consider for example the dynamics described in Figure 2.5.

The rates satisfy $r_A + r_B + r_C = 1$ and $\tilde{r}_A + \tilde{r}_B + \tilde{r}_C = 1$, and can be interpreted as density reservoirs. For this model the hydrodynamic equation is similar to (8) and it is supplemented with boundary conditions that can be of Dirichlet type or Robin type [39].

Interface models

The models: Now we describe another collection of models with two conservation laws. These systems were introduced in [11], they consist of perturbations of Hamiltonian

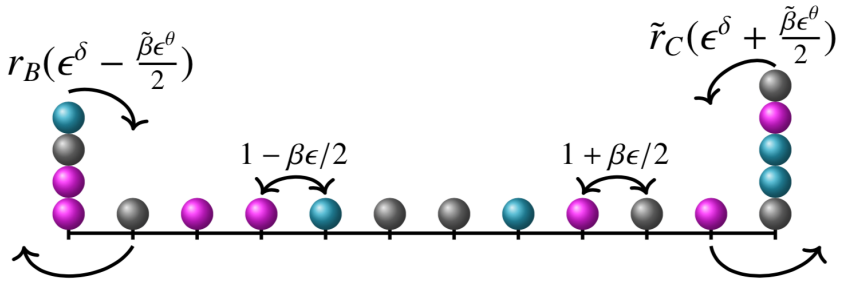


Figure 2.5: Dynamics of the ABC model with reservoirs at $x = 0, 1$. Particles of species A, B and C.

dynamics with a conservative noise and present strong analogies with the standard chains of oscillators. The dynamics of these fluctuating interface models, denoted by $\{\eta_x(t)\}_{t\geq 0}$, depend on some interaction potential $V : \mathbb{R} \to [0, +\infty)$ and have state-space $\widehat{\Omega}_\epsilon := \mathbb{R}^{\mathbb{T}_\epsilon}$ (these variables now are continuous and unbounded). The dynamics conserve two quantities:

$$\text{energy} \sum_x V(\eta_x), \quad \text{and volume} \sum_x \eta_x,$$

and in [11] it was proved that those are the only conserved quantities. Here η_x represents the height of the interface at the site x.

There are some potentials that have been explored in the literature and below we focus on two of them, namely: the exponential potential and the quadratic potential, see[1, 5, 6, 7, 8]. Fix a positive real parameter $b > 0$ and define the Kac-van Moerbeke potential $V_b : \mathbb{R} \to [0, +\infty)$ by

$$V_b(u) = e^{-bu} - 1 + bu.$$

The infinitesimal generator is given by

$$\widehat{\mathcal{L}} = \alpha\epsilon^\kappa \mathcal{A}_b + \gamma\mathcal{S} \tag{9}$$

where $\gamma, \kappa > 0$, $\alpha \in \mathbb{R}$ and $\mathcal{A}_b$ and $\mathcal{S}$ act on differential functions f as

$$(\mathcal{A}_b f)(\eta) = \sum_{x\in\mathbb{T}_\epsilon} \big(V_b'(\eta_{x+\epsilon}) - V_b'(\eta_{x-\epsilon})\big)(\partial_{\eta_x} f)(\eta) \tag{10}$$

$$(\mathcal{S} f)(\eta) = \sum_{x\in\mathbb{T}_\epsilon} (f(\eta^{x,x+\epsilon}) - f(\eta)). \tag{11}$$

The configuration $\eta^{x,x+\epsilon}$ represents the swapping of particles as described above. For more details on the definition of these models we refer to [5, 11, 53]. The parameter $\alpha\epsilon^\kappa$ regulates the intensity of the Hamiltonian dynamics in the system in terms of the scaling parameter ϵ. The role of the parameter γ is to regulate the intensity of the stochastic noise. We note that when $\gamma = 0$ (i.e. absence of noise) this system is completely integrable. We will speed it up in the time scale $t\epsilon^{-a}$ with $a > 0$.

As mentioned above, the system has two conserved quantities: energy $\sum V_b(\eta_x)$ and volume $\sum \eta_x$, but of course, since the generator is a linear operator, any linear combination (plus constants) of energy and volume is also conserved, as e.g. $\sum_x \xi_x$ with $\xi_x = V_b'(\eta_x)$. Now we describe the space-time evolution of the relevant quantities of the system.

Hydrodynamic limit for interface models: We define the empirical measures associated with the energy and the volume as in (2) by

$$\begin{cases} \pi^{\epsilon,e}(\eta, du) = \epsilon \sum_{x\in\mathbb{T}_\epsilon} V_b(\eta_x)\, \delta_x\,(du) \\ \pi^{\epsilon,v}(\eta, du) = \epsilon \sum_{x\in\mathbb{T}_\epsilon} \eta_x\, \delta_x\,(du)\,. \end{cases} \tag{12}$$

In [11], for $a = 1$ and in the strong asymmetric regime it was proved (before the shocks) that the hydrodynamic equations (of hyperbolic type) are given by

$$\begin{cases} \partial_t e - \alpha b^2\, \nabla(e - bv)^2 = 0 \\ \partial_t v + 2\alpha b\, \nabla(e - bv) = 0. \end{cases} \tag{13}$$

As for the ABC model, the hydrodynamics is given by a system of coupled equations, but instead of parabolic equations, here we have hyperbolic equations.

We conclude by noting that for models described above, we obtained a variety of PDEs with several boundary conditions. The exploration of other types of boundary conditions and more general PDEs is certainly important and deserves attention. Moreover, we believe that with the knowledge of the underlying SIPS we can get information on the notion of weak solutions to some PDEs in a probabilistic way.

3 Equilibrium fluctuations

In the last section we have analysed a Law of Large Numbers for the empirical measure in SIPS with one or more conservation laws. The limit in the hydrodynamic limit is deterministic and we know what is the typical profile that we should observe at any time t. The question that we can address now is related to the corresponding Central Limit Theorem, i.e. establishing the description of the fluctuations around the hydrodynamic limit. Typically, the study of non-equilibrium fluctuations is very intricate since it requires a deep knowledge about the correlations of variables and this can be quite difficult for the majority of the dynamics. What one is searching for, in the equilibrium scenario in e.g. exclusion processes, is the fluctuations around the constant hydrodynamical profile, see the Figure 3.1.

We start by describing what can happen for systems with one conservation law and then we consider the case of more conservation laws.

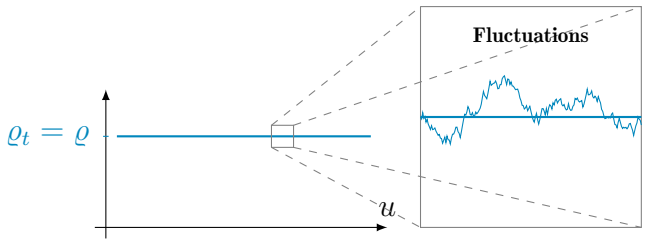


Figure 3.1: Fluctuations around the typical behavior.

3.1 Fluctuations for systems with one-conservation law: the exclusion process

As above, first we focus on a system with one conservation law, the exclusion process, and from now on, we assume that it starts from the stationary state, the Bernoulli product measure of parameter $\varrho \in (0,1)$ given in (2). We define the empirical field associated to the density, which is the linear functional defined on $f : \Lambda \to \mathbb{R}$ (belonging to a suitable space) as

$$\mathcal{Y}_t^\epsilon(f) = \sqrt{\epsilon} \sum_{x \in \Lambda_\epsilon} f(x)(\eta_x(t\epsilon^{-a}) - \varrho). \tag{14}$$

The last identity is obtained by first integrating the test function f with respect to the density empirical measure in (2), then removing the mean with respect to (2), and then dividing it by $\sqrt{\epsilon}$. The question that arises now is to understand the limit in distribution, as $\epsilon \to 0$, of $\mathcal{Y}_t^\epsilon$, denoted by $\mathcal{Y}_t$. For the exclusion processes introduced above, one can get several different limits:

A. for the SSEP and in diffusive scaling $a = 2$, the Ornstein-Uhlenbeck (OU) process given by

$$d\mathcal{Y}_t = 1/2\Delta\mathcal{Y}_t dt + \sqrt{F(\varrho)}\nabla\dot{\mathcal{W}}_t. \tag{15}$$

B. for WASEP with a weak asymmetry i.e. $\kappa > 1/2$ and in the diffusive scaling $a = 2$, the same as (15), while for $\kappa = 1/2$ the KPZ equation (introduced in [44] or its companion, namely the stochastic Burgers (SB) equation, respectively, for the height field h_t or for the density field $\mathcal{Y}_t$:

$$\begin{aligned} dh_t &= 1/2\Delta h_t dt + 4E(\nabla h_t)^2 dt + \sqrt{F(\varrho)}\dot{\mathcal{W}}_t \\ d\mathcal{Y}_t &= 1/2\Delta\mathcal{Y}_t dt + 4E\nabla\mathcal{Y}_t^2 dt + \sqrt{F(\varrho)}\nabla\dot{\mathcal{W}}_t. \end{aligned} \tag{16}$$

Above we used the notation $\dot{\mathcal{W}}_t$ for the standard space-time white-noise.

The height field can be defined analogously to the density field, but the relevant quantity for this field is the net flux of particles $J_{x,x+1}$ through the bond $\{x, x+1\}$, and the definition of the field is as in (14) but replacing η_x and its average by $J_{x,x+1}$ and the corresponding average.

The results described above were analyzed in [18, 31, 32, 33, 36, 37, 40, 41] and were extended to many other models in stationarity, but recently, some of them have been extended to the non-equilibrium scenario, see [55].

C. for ASEP, i.e. $E = 0$, $p \neq 1/2$ and in hyperbolic scaling $a = 1$,

$$d\mathcal{Y}_t = (1 - 2\varrho)(1 - 2p)\nabla\mathcal{Y}_t dt.$$

Observe that in the last identify, if we consider $\varrho = 1/2$ we get a trivial evolution for the density field. The same result holds if instead, we redefine the field in a frame with the velocity $(1 - 2\varrho)\epsilon^{1-a}$. Therefore, to get a non-trivial behavior we have to speed up the time and for the choice $a = 3/2$ the limiting field should be given in terms of the so-called KPZ fixed point, which has been constructed in [48]. In [30] it was proved that up to the time scale $t\epsilon^{4/3}$ there is no evolution of the density field, and its law coincides with the law of the initial field $\mathcal{Y}_0$. Nevertheless, beyond that time scale it is not known yet the limit, but it should be given in terms of the KPZ fixed point. The results of [30] applied to WASEP show that below the line $a = 4/3(\kappa + 1)$ there is no time evolution, but in fact, the trivial evolution should go up to the line $a = 3/2(\kappa + 1)$, see the gray region on Figure 3.2.

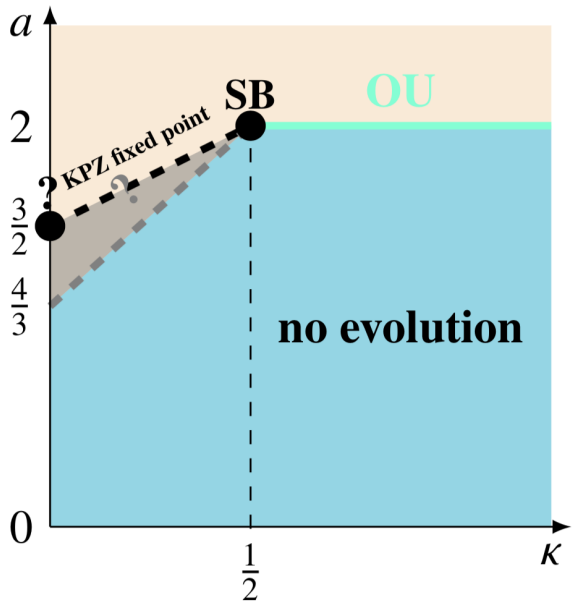


Figure 3.2: Fluctuations of the density in WASEP.

For a transition probability allowing long jumps, the limit behaviour can be Gaussian, or given in terms of a fractional OU (when the symmetry dominates) or the fractional SB equation (when symmetry and asymmetry have exactly the same strength), see [34, 35].

In the case of exclusion processes given by a general transition probability, we have already seen possible laws, given as solutions to stochastic PDEs (SPDEs) ruling the fluctuations of the unique conserved quantity, the number of particles. The way to connect

one solution to the other could be either by changing the nature of the tail of the transition probability or the symmetry/asymmetry dominance phase of the transition probability. The nature of the SPDE is very much related to the underlying SIPS, but the same equation can be obtained from a variety of different particle models and, in that sense, it is universal.

3.2 Fluctuations for multi-component systems

We observe that the results described in last subsection are for systems with (only) one conservation law and for these there is no ambiguity on the choice of the fields that one should look at, it is the field associated to the conserved quantity. When systems have more than one conserved quantity, and whose evolution is coupled, as is the case for the ABC model or the interface models that we described above, we have to be careful when we define those fields. Moreover, a special feature of multi-component models is the fact that different time scales coexist and this never occurs for systems with only one conserved quantity.

In [52], with a focus on anharmonic chains of oscillators, it was developed the non-linear fluctuating hydrodynamics theory (NLFH) for the equilibrium time-correlations of the conserved quantities of that model and analytical predictions were done based on a mode-coupling approximation. Roughly speaking, Spohn's approach starts at the macroscopic level, i.e. assumes that a hyperbolic system of conservation laws is governing the macroscopic evolution of the empirical conserved quantities. Then adds a diffusion term and a dissipation term to the system of coupled PDEs and then linearizes the system at second order with respect to the equilibrium averages of the conserved quantities. A fundamental role is played by the normal modes, i.e. the eigenvectors of the linearized equation and these modes evolve with different velocities and in different time scales. These modes might be described by different forms of super-diffusion or standard diffusion processes and this description depends on the value of certain coupling constants. From this approach, many other universality classes pop up, besides the Gaussian or the KPZ, already seen in systems with only one conservation law. Despite all the complications that one might face when dealing with multi-component systems, there is a choice of the potential V for the interface models described above, for which all the diagram for the fluctuations of its conserved quantities has been obtained. Now we quickly describe it.

The harmonic potential: Let us consider now the generator given in (9) but with the quadratic potential $V(x) = \frac{x^2}{2}$, the harmonic potential. The invariant measures $\mu_{v,\beta}$ are explicitly given by

$$\mu_{v,\beta}(d\eta) = \prod_{x\in\Lambda_\epsilon} \Big(\frac{\beta}{2\pi}\Big)^{1/2} \exp\Big\{ -\frac{\beta}{2}\Big(\eta_x - v\Big)^2\Big\} d\eta_x, \tag{17}$$

where $v \in \mathbb{R}$ and $\beta > 0$. In this case the system conserves two quantities, the energy and the volume: $\sum_x \eta_x^2$ and $\sum_x \eta_x$ and note that the average with respect to $\mu_{v,\beta}$ of η_x and η_x^2 is equal to v and $v^2 + \frac{1}{\beta}$, respectively. According to NLFH, the quantities that one

should analyse are now given by

$$\mathcal{U}_1 = \overline{\eta}_x \quad \text{and} \quad \mathcal{U}_2 = 2v\overline{\eta}_x + \overline{\eta_x^2}.$$

For a random variable X, the notation $\bar{X}$ denotes the centered random variable. Note that for $v = 0$ we simply get $(\mathcal{U}_1, \mathcal{U}_2)$ as the volume and energy. The corresponding fields should be taken on a frame with velocity $v_1 := 2\alpha_\epsilon$ and $v_2 := 0$. According to NLFH, in the strong asymmetric regime ($\kappa = 0$) $\mathcal{U}_1$ should behave diffusively and $\mathcal{U}_2$ should behave as a Lévy process with exponent $\frac{3}{2}$. For the volume, i.e. the quantity $\mathcal{U}_1$, when we take the fluctuation field with velocity equal to 0, then we get a process that is linearly transported in time, see the line in light blue colour in Figure 3.3, while if we take it with the velocity v_1 we get an OU process without drift, see the line in magenta colour. For $\mathcal{U}_2$

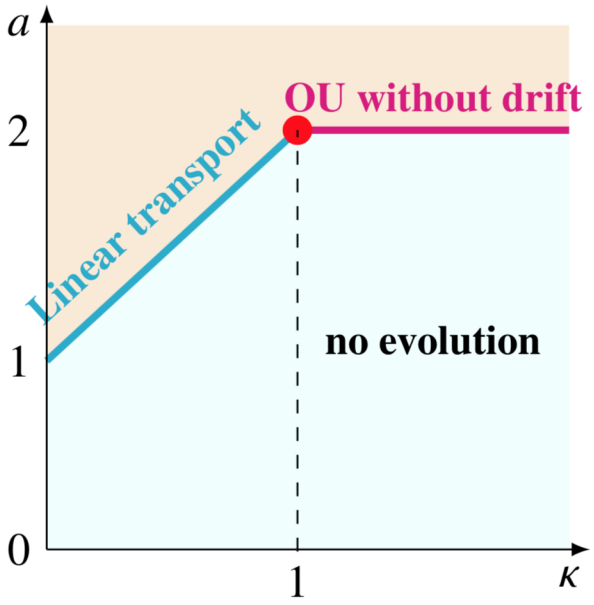


Figure 3.3: Fluctuations for $\mathcal{U}_1$

with velocity $v = 0$, i.e. the energy (recall that $v_2 = 0$) we have the results summarized in Figure 3.4.

In Figure 3.3, the line in light blue colour corresponds to $a = \kappa + 1$, while the line in purple colour in Figure 3.4, where we see the Lévy process with exponent $\frac{3}{2}$, corresponds to $a = \frac{3}{2}(\kappa + 1)$. We note that this diagram is complete, but the method that was employed to derive these results heavily relies on the specific form of the dynamics.

There is still much work to do in this direction and we believe that one should analyse the action of the generator in other relevant quantities and keep track of those that give a non-trivial contribution to the limit. There are several equations that one can obtain

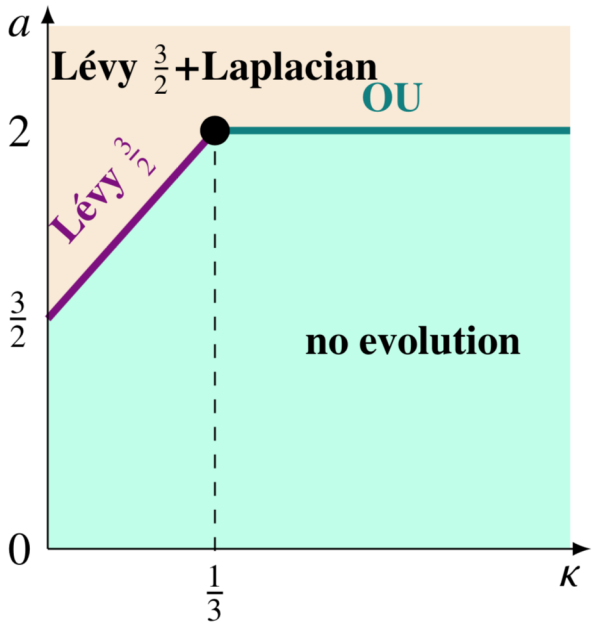


Figure 3.4: Fluctuations for $\mathcal{U}_2$

from this procedure by using many different microscopic dynamics and for this reason they are said to be universal. Understanding how to connect universality classes is a major problem in the field of SIPS. There is much to do regarding this problem and in the next years hopefully there will be large steps in this direction.

4 Final comments

Some of the problems described above, were part of the goals of the research project with acronym HyLEF, *Hydrodynamic Limits and Equilibrium Fluctuations: universality from stochastic systems* and it is one of the projects funded by the European Research Council (ERC) in the 2016 edition of the ERC Starting Grants. Until now, this is the first and so far the only ERC grant awarded in Portugal in the field of Mathematics and it is headed by the author of this article, Patrícia Gonçalves, now a Full Professor at the Mathematics Department of Instituto Superior Técnico (IST) of the University of Lisbon. This is a grant of nearly 1.2 million euros for 5 years (postponed to 7 years due to the pandemic period) which started on the 1st December 2016.

The budget allowed creating a team composed by 4 post-doctoral researchers (2 years each), 2 Ph.D. students (4 years each) and 2 master students (1 year each). This was the first team in Portugal working on the field of SIPS. The budget also allowed organizing conferences and inviting external collaborators to work with the team at IST in Portugal.

I would like to thank to the ERC and to all the members of the Panel PE1 (Mathematics), that by selecting my project for funding, they have all contributed to a big change in my life and in all the people involved in this project. If HyLEF was not funded

by the ERC, the creation of this team under national funds would have been completely impossible.

The group of collaborators of this project include several researchers, some of them working at the host institution and others working abroad, mainly at IMPA and at the Universities of Arizona, Juelich, Lyon, Nice, among others. Below is the picture of some of these members, to whom I am truly grateful for making the last years at IST extremely exciting, not only research wise but also personally. I will certainly remember them for a long time.

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The VIASM-IAMP Summer School in Mathematical Physics, Quy Nhon, 2023

by Phan Thành Nam and Toan T. Nguyen

The VIASM-IAMP Summer School in Mathematical Physics took place in Quy Nhon during August 1-5, 2023. This activity was organized by the Vietnam Institute for Advanced Study in Mathematics (VIASM) in cooperation with Quy Nhon University (QNU) and the International Center for Interdisciplinary Science and Education (ICISE), and also partially sponsored by the IAMP. The goal of the summer school was to initiate a long-term program to promote the development of mathematical physics in Vietnam and other countries in Asia.

The scientific program of the summer school was put together by us, while the local organizers included Minh Ha Le (VIASM), Cong Trinh Le (QNU) and Thanh Son Tran (QNU & ICISE). It attracted 80 onsite participants, including students in mathematics and physics, young researchers and experts, from Vietnam, the US, Italy, Germany, Japan, Korea, and China.

The summer school had 4 minicourses (4 hours each) at QNU given by:

- John R. L. Anderson (Stanford): Nonlinear Waves in fluids and general relativity
- Nguyen Viet Dang (Paris Sorbonne): Some aspects of stochastic quantization
- Mathieu Lewin (Paris Dauphine): Statistical mechanics of Coulomb and Riesz gasses
- Luc Nguyen (Oxford) : Analysis of topological defects in nematic liquid crystals

These minicourses featured a wide range of selected topics, introducing several current research directions in classical and quantum mechanics as well as related mathematical questions in spectral theory, partial differential equations, geometry and probability.

On August 3, Sergiu Klainerman (Princeton) gave a public lecture at QNU addressing questions like “Is mathematics unreasonably effective? Why?” and “Are black holes real?”. The lecture attracted about 300 onsite participants, including 33 high school students, and 100 online participants.

Minicourses and public lecture at QNU

The summer school was closed by a one-day workshop at the ICISE, with one-hour lectures of six young researchers: Alejandra Ramirez Luna (Santa Cruz), Konstantin Merz (Osaka), Quoc-Anh Ngo (VNU Hanoi), Trinh T. Nguyen (Wisconsin-Madison), Dinh-Thi Nguyen (Uppsala), and Minh-Phuong Tran (Ton Duc Thang University). All participants were also invited to a half-day tour to visit a few historical attractions in Quy Nhon.

Workshop participants at ICISE

Further information of the VIASM-IAMP Summer School in Mathematical Physics 2023 can be found at https://viasm.edu.vn/hdkh/iamp2023.

Given the success of the summer school, the VIASM plans to continue organizing many activities promoting mathematical physics. The next summer school has been scheduled in Hue, August 5-10, 2024.

Announcements

Third IST Austria Summer School in Analysis and Mathematical Physics

We would like to draw your attention to the 3rd IST Austria Summer School in Analysis and Mathematical Physics, to be held June 10-14, 2024 at IST Austria in Klosterneuburg on the outskirts of Vienna. It is the third summer school in a bi-annual series on analysis and related areas held at IST Austria. The event is aimed at a broad audience of young researchers in these fields, particularly focusing on PhD students and postdocs.

The summer school will consist of three main courses as well as a number of additional invited talks. The main courses will be given by

- Maria Colombo, EPFL Lausanne - "Nonuniqueness and anomalous dissipation in the Navier-Stokes equations"
- Mathieu Lewin, Universite Paris Dauphine - "Mathematical foundations of Density Functional Theory"
- Jan Philip Solovej, University of Copenhagen - "The universal structure of large atoms and molecules"

Registration is free but mandatory. A limited amount of financial support for accommodation and/or travel costs is available; for application for financial support, a letter of motivation and a CV should be submitted along with the registration. The deadline for application for financial support is March 31, 2024; the registration deadline is May 15, 2024.

Detailed information on the summer school and the registration can be found at https://summerschool-analysis.ist.ac.at/.

We hope for a widespread participation in our summer school!

With best regards,

The Organizers
(Laszlo Erdős, Julian Fischer, Jan Maas, and Robert Seiringer)

International Colloquium on Group Theory Methods in Physics
Group33/35, Cotonou, Benin, July 15-19, 2024
(https://icgtmp.sciencesconf.org/)

The series of International Colloquia on Group Theoretical Methods in Physics was initiated in 1972 following the efforts of Henri Bacry in Marseille and Aloysio Janner in Nijmegen (https://icgtmp.blogs.uva.es). The aim was to provide a forum for physicists interested in group-theoretical methods which were mostly represented at that time by various communities: elementary particle theoreticians and phenomenologists, nuclear and atomic physicists and solid state specialists. Also mathematicians were eager to apply newly developed group theoretical techniques and algebraic structures.

The annual Colloquia developed in an international series in the field of group theory in its widest sense and have earned a strong scientific reputation around the world. It has become a bi-annual Colloquium since 1990 (Group18 Moscow). The list of the venues and corresponding proceedings confirms the claim of geographical and scientific diversity. The last edition took place in Strasbourg from July 18 to 22, 2022 (https://indico.in2p3.fr/event/23498/).

Over the years, it has further broadened and diversified due to the successful application of group theoretical, geometric and algebraic methods in life sciences and other areas. The conference has an interdisciplinary character. It aims at bringing together experts and young researchers from different fields encouraging cross disciplinary interactions. Currently the ICGTMP attract theoretical and experimental physicists, mathematicians and scientists in related disciplines who are interested in the latest methods and applications of group theory in its widest sense.

The ICGTMP Standing Committee is the organizing body of the series of colloquia which oversees and maintains the continuity of the series. It places equal importance on the interest of physicists to approach physical problems using novel mathematical settings and the interest of mathematicians to construct new mathematical frameworks with reference to physical knowledge and intuition. A. Janner was the first chairman of the Standing Committee, acting from 1972 to 1986. He was succeeded by L. Biedenharn from 1987 to 1993, H.-D. Doebner from 1994 to 2008, J.-P. Gazeau from 2009 to 2014, and Mariano A. del Olmo from 2014 to the present.

During his time as chairman, Heinz-Dietrich Doebner convinced the Standing Committee that it would be necessary for the future development of our field to acknowledge young researchers who presented outstanding work and to motivate them, to continue and to diversify their activity. He proposed to award in each Colloquium a prize. The first Hermann Weyl Prize was awarded in 2002 to Edward Frenkel.

In 2021 the ICGTMP Standing Committee organised a new award called Weyl-Wigner Award to replace the Wigner Medal. The purpose of this new Award is to recognize outstanding contributions to the understanding of physics through group theory, continuing the tradition of The Wigner Medal (1978-2018). The first Weyl-Wigner Award was awarded in 2022 in Strasbourg to Nikolai Reshetikhin.

Africa will host this Colloquium for the first time in 2024 in Benin, an edition initially planned to be held in 2020 (Group33), but which was postponed due to the COVID-19 pandemic. Similar meetings are routinely held in Europe and North America, but financial resource limitations restrict their access to African students and researchers drastically. On the other hand, holding such a meeting at an African institution immediately makes it accessible to a large number of Africans. In this manner, the impact of Group33/35 edition is and will definitely be significant.

Meeting on "Random Matrices and Integrability in Complex and Quantum Systems" postponed

Regrettably, international tensions have impacted some meetings that have been announced through IAMP. In particular, the organizers of the workshop "Random Matrices and Integrability in Complex and Quantum Systems," originally scheduled for October 25–30, 2023 in Yad Hashmona, Judean Hills, Israel, has been postponed to June 26–July 01, 2024. For currrent information visit this site.

Time's Arrow

Scientific anniversaries

1923. On 10 September Louis de Broglie introduced the notion of the wave-particle duality in "Ondes et quanta," presented at the Paris Academy of Sciences.

1923. On 1 November Arthur Compton published experiments confirming his scattering formula in The Physical Review.

1923. On 15 December Freeman Dyson was born in Crowthorne, Berkshire, England.

1923. Robert Millikan received the Nobel Prize "for his work on the elementary charge of electricity and on the photoelectric effect."

1973. Brian Josephson received the Nobel Prize in Physics "for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effects"

1998. Google was founded on 4 September.

Personal celebrations

Spectral Analysis and Applications, a conference in honor of Peter Kuchment's scientific work, October 13-15, 2023, at Texas A&M University, College Station, USA.

Lost luminaries

Viktor P. Maslov, 3 August, 2023.

Mary Beth Ruskai, 27 September, 2023.

Vladimir E. Zakharov, 20 August, 2023.

Readers are encouraged to send items for "Time's Arrow" to bulletin@iamp.org.

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. Rahul Hingorani, University of California, Davis, USA
2. Abdul Rahimyar, University of Stony Brook, USA
3. Thomas Jackson, University of California, Davis, USA
4. Professor Valter Moretti, University of Trento, Italy
5. Doctor Nicolò Drago, University of Trento, Italy
6. Nicolai Rothe, Technical University Berlin, Germany
7. Doctor Min Chul Lee, University of Oxford, UK
8. Doctor Thao Thuan Vu Ho, Monash University, Melbourne, Australia
9. Doctor Daniel Jaud, Germany
10. Dr. Nathan Clisby, Swinburne University of Technology, Hawthorn, Australia

Recent conference announcements

Integrable Systems, Random Matrices and Special Functions at UBI

November 20, 2023, at University of Beira Interior, Covilha, Portugal.

TexAMP 2023/24 Texas Analysis and Mathematical Physics Symposium

February 9-11, 2024, at Texas A&M University, College Station, Texas, USA.

For an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

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