International Association of Mathematical Physics

News Bulletin

October 2010

International Association of Mathematical Physics News Bulletin, October 2010

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Cover photo: The IAMP members Stas Smirnov and Cédric Villani (photo by courtesy of Pierre Maraval) are Fields Medals laureates 2010.

On Fields, percolation and damping

by Pavel Exner (IAMP President)

In science as in life, some communities enjoy an island-like status while others are surrounded by neighbours. The bigger these are the more attention you have to pay to them. This is certainly the situation in our field finding itself between two big disciplines, both having traditions much longer than any political and cultural entities in which they had been cultivated.

An important event which in a four-year cycle attracts our attention to mathematics is the International Congress of Mathematicians. After the last twelve-year conjunction in 2006 it is now out of phase with our congress; this year ICM convened in August in Hyderabad, being attended by about three thousand people, and the next one is scheduled for Seoul in 2014.

The ICM opening is traditionally expected with impatience to see whom the International Mathematical Union will distinguish with their highest awards. This year was fortunately free of sensation-seeking journalists but what is more important for us, it brought us great joy when two out of four Fields Medals went to IAMP members. The last time such a thing happened was twenty years ago when Fields went to Vaughan Jones and Edward Witten.

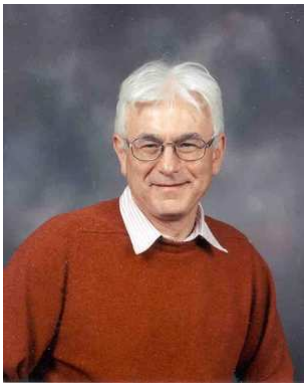
The two latest laureates are *Stanislav Smirnov*, distinguished for "for the proof of conformal invariance of percolation and the planar Ising model in statistical physics", and *Cédric Villani*, whose citation says "for his proofs of nonlinear Landau damping and convergence to equilibrium for the Boltzmann equation". We congratulate both of them from all our heart and we devote a part of this News Bulletin issue to their work.

Let me also recall that Cédric was among the Henri Poincaré Prize laureates at the last-year ICMP; the only two other IAMP members with such a double distinction are Maxim Kontsevich and Edward Witten. Let me also mention that it is the second Fields medal for percolation-related problems after the one won four years ago by Wendelin Werner.

One pleasant aspect in all that which I would like to stress is that the mathematical community appreciated significantly work on problems which not only come from physics but they are, in one way or another, a subject of current interest among physicists. Some of our members complain that the link between the two disciplines on which our community thrived is getting weaker recently. Let us regard this-year Fields medals as a good omen.

Stas Smirnov received the Fields Medal

by Yuri Suhov (Cambridge, UK)



1. An informal laudatio

The award of the Fields medal to Stanislav Smirnov at the last ICM in Hyderabad in August, 2010, has a high significance for the Mathematical Physics community. This was of course the second Fields medal in a row (after Wendelin Werner's in 2006 in Madrid) for the work in an area where Probability joins Complex Analysis and together they successfully attack previously impenetrable-looking problems, in particular from Theoretical Physics. For me, the probabilistic flank of this attacking force is commanded by Wendelin (and the wise advisor Greg Lawler), but the analytic flank is definitely led by Stanislav. It is a great pity that the other general, Oded Schramm, the visionary who dared to launch this glorious campaign and left a substantial legacy, is no longer in the attacking rows.

In a wider context, Smirnov's medal should be considered as an extraordinary achievement of a relatively small group of people (most of whom have no physical background) who tirelessly push forward our rigorous knowledge in this area. We should bear in mind that until less than 15 years ago, the above-mentioned problems admitted no proper mathematical formulation, let alone any hope of their rigorous (and comprehensive) solution. At least this was my impression when I listened to Sasha Polyakov at the Dobrushin–Malyshev—Minlos–Sinai Math Phys seminar back in Moscow. I remember how in the 1980's, during Polyakov's aspiring presentation on conformality in the 2D Ising model, Roland Dobrushin tried to take notes (which was very atypical of him). But the whole style of the Polyakov's talk was so different from a re-fined mathematical machinery (which of course defines an ideal style of a scientific talk for a mathematician) that in the end Dobrushin gave up. He sighed deeply and scribbled a sentence: "A concerto by a lyrical physicist." When he saw me looking over his shoulder, he smiled and whispered: "How many decades will it take to convert this into science?".

Before I pass to Smirnov's works, I would like to mention another aspect of his research personality: he is a true patriot of St. Petersburg (and probably will remain forever, no matter where he resides or travels). I think he firmly believes that Moscow is the capital city only in the sense of concentration of bureaucracy. In mathematical research, St.Petersburg community often prides itself on delicate analytic work for which vane Moscovites, craving a quick success, have a little time. During the Soviet period, the Analysis group in Leningrad kept mainly to themselves (comparatively speaking, of course), and their achievements and influence became widely known and acknowledged only later. In any case, many places of Smirnov's works recall of what St.Petersburg is famous for: a deep and broad knowledge of mathematical classics, elegant and straightforward ideas, meticulous analytics and last, but not least, a strong focus on the ultimate

goal.

Smirnov's work of 2001-2003 on conformality in percolation (cited in the Fields Medal Committee decision) has been already commented on in several places. I therefore feel it is appropriate to comment briefly on a more recent theorem, on conformality in the 2D Ising model. I'll say from the start that the result will be about the random curve Γ separating two phases in a domain Λ under the so-called Dobrushin boundary conditions on $\partial\Lambda$. More precisely, it will be about the probability that Γ passes through (or near) a given point $z \in \mathbb{C}$. Even more precisely, the expected value of the imaginary exponent of a certain 'winding' random number defined for Γ, as a function of z, exhibits conformality in the limit when the lattice mesh tends to 0.

I will only discuss properties of the separation line emerging in the so-called random cluster (RC), or Fortuin-Kastelein (FK), representation of the Ising model: the whole analysis in this case is more accessible than for the curve separating the $\pm$ spins. See reference [S] at the end of this note.

2. Random-cluster separation curve

In the RC representation, the 2D Ising model emerges as a 'projection' of a more intricate model living on two mutually dual square lattices $\mathbb{Z}^2$ obtained from each other by a shift by vector $(1/2; 1/2)$ (edges of one lattice pass, orthogonally, through the mid-points of the edges of the other). In addition, we will need a *medial* lattice whose vertices are the mid-points of the edges of the original lattice (primal and/or dual). The edges of the medial lattice join its nearest neghboring vertices; if the edge-length in the primal and dual lattices equals 1, then the edge-length of the medial lattice is $1/\sqrt{2}$. The medial lattice is again square but rotated by the angle $\pi/4$.

An RC configuration is specified when some of the edges of one of the two lattices $\mathbb{Z}^2$ (call this lattice primal and denote it by $\mathbb{Z}^2_{\rm prim}$) are removed: they are declared closed while the the remaining edges by $\mathbb{Z}^2_{\rm prim}$ are called open. The associated dual configuration lives on the dual lattice $\mathbb{Z}^2_{\rm dual}$; it is determined by requiring that a dual edge orthogonal to a primal open one is closed and vice versa. Next, a (primal or dual) cluster is defined as a maximal subset of the primal/dual lattice connected by open edges. Each pair of primal/dual configurations is determined by a corresponding collection of *loops* drawn on the 'mollified' medial lattice $\mathbb{Z}^2_{\rm med}$ (with smoothened corners) and dividing the primal clusters from the dual ones. See Figure 1 below.

Thus, the Ising spin probability distribution is generated by an ensemble of random clusters, or, equivalently, by an ensemble of dividing loops on the medial lattice. The starting observation for Smirnov's work is that at criticality, the statistical weight of a loop configuration is

$$\propto \left(\sqrt{2}\right)^{(\text{number of loops})}. \qquad (1)$$

(This is what happens to the RC representation of the standard spin 1/2 Ising model in two dimensions, at the zero magnetic field and the critical temperature.) In fact, Eqn (1) (and the above specification of how exactly the loops and the separating line are drawn) is *all* that we need from the RC Ising model !

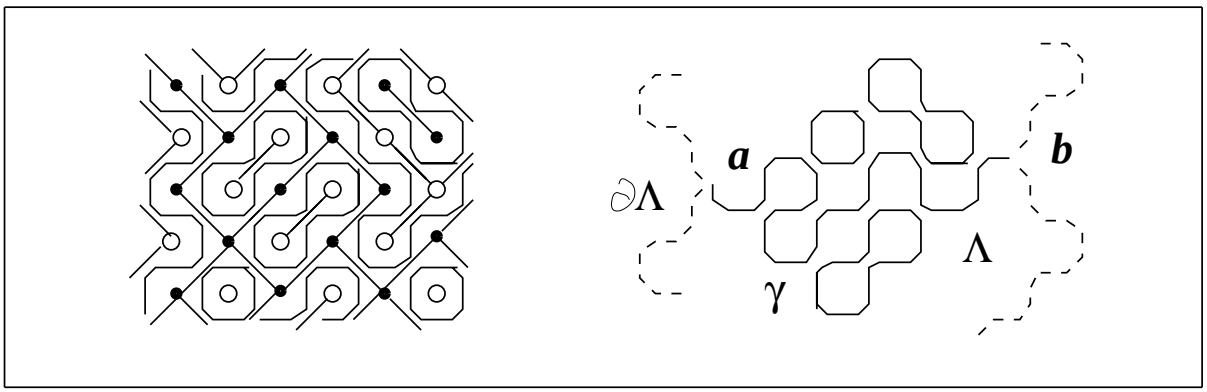


Figure 1.

In this context, the Dobrushin boundary conditions emerge when we take a closed 'contour' $\partial\Lambda$ that serves as the boundary of a 'lattice domain' Λ, and a pair of points a, b drawn in such a way that on one half of $\partial\Lambda$ between points a and b we can put the *primal open* boundary condition and on the other half the *dual open* one. This imposes some requirements upon how exactly $\partial\Lambda$ should be drawn and where the points a and b are placed (relative to lattices $\mathbb{Z}^2_{\text{prim}}$, $\mathbb{Z}^2_{\text{dual}}$ and $\mathbb{Z}^2_{\text{med}}$), but it all can be done in an unambiguous way.

Then the aforementioned boundary conditions generate an ensemble consisting of (i) a random separating curve Γ and (ii) a random collection of loops on both sides of Γ. The statistical weight is given by Eqn (1). A sample of the random curve Γ is denoted by γ. See Figure 1.

In addition, we have to assume that the boundary $\partial\Lambda$ is 'wiggling' in the sense that the neighboring boundary edges are orthogonal. (This is a necessary technical assumption.)

Finally, take all possible RC configurations in Λ, or - equivalently - configurations of loops on edges of the medial lattice $\mathbb{Z}^2_{\text{med}}$ inside Λ, compatible with the open boundary conditions as in (iii). Introducing the weight of a configuration as in Eqn (1), we obtain the probability distribution for the separating line Γ. This probability distribution is denoted by $P_{\Lambda,a,b}$ and the expectation under $P_{\Lambda,a,b}$ by $E_{\Lambda,a,b}$.

All this stuff can be done for a mesh δ (with the edge-length in the medial lattice $\delta/\sqrt{2}$), with the incorporated notation $E^{\delta}_{\Lambda,a,b}$.

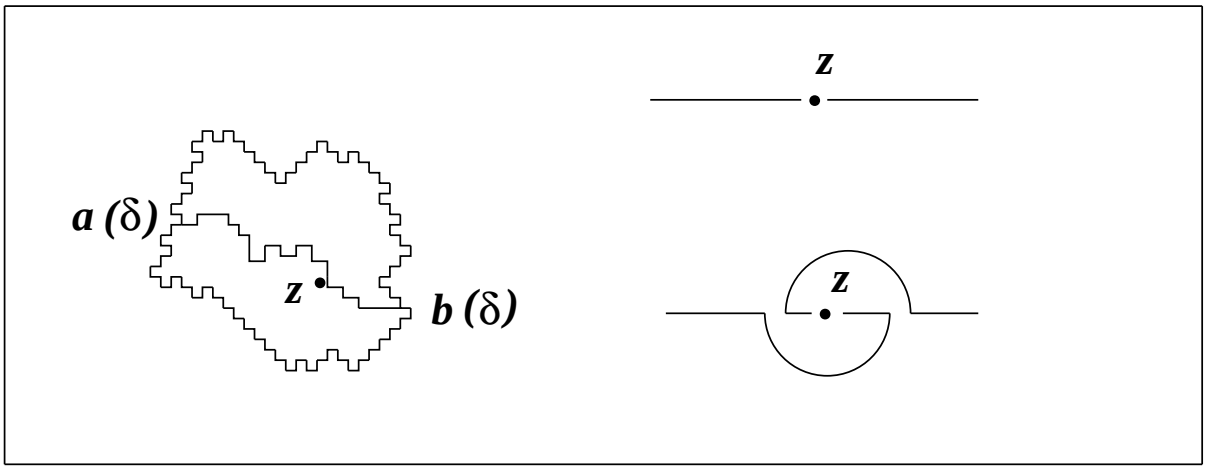


Figure 2.

3. Random-cluster conformality in the 2D Ising model

It is convenient to begin with a domain D in the complex plane $\mathbb{C}$, with two fixed distinct points $x, y \in \partial\Lambda$. Then, given a (small enough) $\delta > 0$, we can construct an 'approximating' lattice domain $\Lambda(\delta)$ and specify points $a(\delta)$ and $b(\delta)$ approximating x and y:

$$\Lambda(\delta) \to D,\ a(\delta) \to x,\ b(\delta) \to y. \tag{2}$$

Under the Dobrushin boundary conditions we obtain an ensemble of separating curves going from $a(\delta)$ to $b(\delta)$ with expectation $E^{\delta}_{\Lambda(\delta),a(\delta),b(\delta)}$.

Smirnov considers the expected value

$$F_\delta(z; D, x, y) = E_{\Lambda(\delta),a(\delta),b(\delta)}\left[\mathbf{1}(z\dot{\in}_\delta\Gamma)\mathrm{e}^{\,\mathrm{i}\,W_\delta(\Gamma,z)}\right],\ \ z \in D. \tag{3}$$

Here $z\dot{\in}_\delta \in \Gamma$ means that a circle $O_\delta(z)$ of radius δ centered at point z has a non-empty intersection with Γ. Furthermore, $W(\Gamma, z)$ is the winding number for curve Γ at point z. Given a sample γ of Γ, $W(\gamma, z)$ is defined as the number of times γ winds around before it enters the circle $O_\delta(z)$. See Figure 2.

The main result of [S] is the following theorem:

As $\delta \to 0$, the function $z \in D \mapsto \dfrac{1}{\delta^{1/2}} F_\delta(z; D, x, y)$ converges to a function

$$z \in D \mapsto F(z; D, x, y). \tag{4}$$

The limiting function has the following conformality property. If Φ is a conformal map taking D to domain $\Phi(D)$ then

$$F(z; D, x, y) = F(\Phi(z); \Phi(D), \Phi(x), \Phi(y)). \tag{4}$$

I'll conclude with a few comments.

1. The stated result suggests that the function $F(z; D, x, y)$ has some built-in connection to a conformal map of D to some "canonical" domain. Indeed, $\Phi(z; D, x, y) = \sqrt{\phi'_{D,x,y}(z)}$, where $z \in \mathbb{C} \mapsto \phi(z)$ is the conformal map from D to a unit strip $\mathbb{R} \times (0, 1)$, taking points x and y to $-\infty + 0 \cdot i$ and $+\infty + 0 \cdot i$, respectively.

2. A closely related matter is that functions $F_\delta(z; D, x, y)$ and $F(z; D, x, y)$ solve Riemann–Hilbert problems.

3. An important part of the proof is the (suitably chosen) concept of discrete analyticity, for functions defined on a square lattice. (In [S] Smirnov uses the term "preholomorphic function".) This notion allows us to write down and use some *a priori* estimates which guarantee compactness and – eventually – convergence to the limit, of the family of functions $\frac{1}{\delta^{1/2}} F_\delta(z; D, x, y)$ as $\delta \to 0$.

Choosing the right definition here is important. Smirnov's version of discrete analyticity, as introduced and used in [S], implies discrete Cauchy–Riemann equations, but formally requires slightly more.

4. The limiting behavior of the separating curve Γ is of the great interest. Smirnov's approach leads to the identification of the limiting law for Γ as $\mathrm{SLE}_{8/3}$. (The proof of this fact is not given in [S].) On the other hand, the curve separating the $\pm$ spins in the original Ising model (under Dobrushin-type boundary conditions) should have the limiting law $\mathrm{SLE}_{11/8}$: it is 'less wiggling' than the RC-curve.

5. (A personal note.) Going through paper [S] is not an easy ride. (Probably, the same can be said about many papers in the subject of conformality.) However (rather surprisingly), heavy-handed complex analysis does not have a strong presence in the proofs. But Smirnov's text requires very careful reading (and constant re-reading, given a few typos and a couple of omissions). I strongly recommend to complement [S] with the presentational transparencies that Stanislav generously offers on his webpage. I hope this note will encourage more readers to attempt to understand Smirnov's techniques.

[S] Smirnov, S. Ann. Math., **172**:2, 1435–1467 (2010)

Cédric Villani is awarded the Fields Medal 2010

by Eric A. Carlen (Rutgers, USA)

Cédric Villani is an outstanding mathematician whose work is well known to an unusually large portion of the mathematical community since he has made outstanding contributions to an unusually wide range of fields: analysis, probability, calculus of variations, including optimal mass transportation and the proof of optimal functional inequalities, statistical physics and differential geometry. Thus it was no surprise to many that he was awarded a Fields medal at the 2010 International Congress of Mathematicians; still, it is particularly pleasing to many in the mathematical physics community that the citation for his award focuses on his contributions to statistical mechanics, and especially his work in kinetic theory.

In particular, the citation mentions his work [1] with Laurent Desvillettes on the approach of solutions of the spatially inhomogeneous Boltzmann equation to a global Maxwellian equilibrium for initial data that is not necessarily close to equilibrium. The citation also prominently mentions his much more recent joint work [2] with his former student, Clément Mouhot, on Landau damping. Both of these works constitute fundamental contributions to kinetic theory, settling issues that had remained open and even controversial, for decades.

We now briefly introduce the kinetic theory context of these problems. Let $f(t,x,v)$ be a time dependent particle density on the phase space $\Lambda\times\mathbb{R}^3$, where Λ is some domain in $\mathbb{R}^3$. Ignoring the issue of spatial boundary conditions, the Maxwell-Boltzmann equation is

$$\frac{\partial}{\partial t}f = \nabla_x\cdot(vf) + \nabla_v\cdot\left(\frac{F}{m}f\right) + Q(f,f)\,, \tag{1}$$

where F is the force acting on the particles, m is the mass of a particle, and Q is the Boltzmann collision kernel:

$$Q(f,f)(v) = \int_{R^3\times S^2} B(v-v_*,\sigma)(f'f'_* - ff_*)\mathrm{d}\sigma\mathrm{d}v_*\,, \tag{2}$$

$$f = f(v,t),\ f' = f(v',t),\ f_* = f(v_*,t),\ f'_* = f(v'_*,t)\ ,$$
$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|\sigma}{2}\,,\quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|\sigma}{2}\,,\quad \sigma\in S^2\ , \tag{3}$$

and S^2 is the unit sphere in $\mathbb{R}^3$.

The Boltzmann collision kernel accounts for the change in the phase space density f due to collisions between a pair of molecules whose initial velocities are v' and v'_*, and whose final velocities are v and v'. The collisions are taken to conserve energy

and momentum, and (3) gives a parametrization of all of these kinematically possible collisions.

Depending on the force law mediating the interaction between particles, different types of collisions will occur at different rates. The function $B(v - v_*, \sigma)$ in (2) gives the precise quantitative expression of this. If the interaction between particles is given by a radial power-law potential $V(x) = k|x|^{-r}$, then $B(v - v_*, \sigma)$ has a special form: $B(v - v_*, \sigma) = |v - v_*|^s b(\cos\theta)$ where θ is the angle between $v - v_*$ and σ, and s is a power depending on r. For hard sphere (billiard) collisions, r is effectively infinite, and then $s = 1$. Maxwell, in his 1868 foundational paper, noted that for $r = 4$, $s = 0$. Thus, in this case, the collision rate function B depends only on $\cos\theta$, and this simplicity permits a number of calculations to be carried out that are not possible for other radial force laws. Maxwell argued that Nature would likely take advantage of this, and that such an intermolecular force law should be physically correct. This turned out not to be the case, but the model has remained mathematically useful.

The works [1, 2] of Villani cited above refer to two special cases if this very general equation: In the work on Landau damping, which describes a plasma of heavy particles that do not collide, the collision term Q is left out, and the force law is the electrostatic force law generated by the charge distribution:

$$F(x,t) = -e\nabla\Delta^{-1}\int_{\mathbb{R}^3} f(t,x,v)\mathrm{d}v \ ,$$

where e is the particle charge. In this case, with $Q = 0$, and F as above, (1) becomes the Vlasov–Poisson equation. *This equation is time reversible.* This is, if $f(t,x,v)$ is a solution, so is $f(-t,x,-v)$. However, in 1946, Landau made a prediction that solutions of this equation would, nonetheless, exhibit a certain irreversible behavior. Over the following decades, this was investigated by many authors at a linearized level, and whether or not Landau's predictions were valid for the actual non-linear evolution remained open, and controversial. The work of Mouhot and Villani vindicated Landau's predictions in a mathematical tour *de force*.

While I cannot, in the space available, hope to convey any of the beauty in the analysis of the Landau damping problem, it is perhaps possible to do this for the work on the approach to equilibrium for solution of the Boltzmann equation. This time, set the force F to be zero, but keep the collisions, and assume the collision kernel is, say, the one for hard sphere collisions.

Because of the collisions, the solutions are definitely not time reversible; indeed, Boltzmann's famous H-Theorem asserts that the entropy

$$S[f] - \int_\Lambda \int_{\mathbb{R}^3} f\log f \mathrm{d}v\mathrm{d}x$$

is monotone increasing at each t, and *strictly* so unless for each x, the conditional distribution of the velocities v is a Gaussian distribution that is isotropic about its mean. Such densities are called *local Maxwellian densities*. (Boltzmann's H functional is minus the Gibbs entropy S.) Since the entropy is maximized, given the mean and variance (physically, given the total momentum and energy), at the *global Maxwellian density*, i.e., the

spatially uniform local Maxwellian density with the given mean and variance, one might expect that solutions will tend to this equilibrium solution, no matter how far away from this the initial data lies. Proving this, and showing that it does indeed happen with a good estimate on the rate is, however, a mathematically challenging problem that had remained open for many years.

One of the fundamental results on which the Desvillettes-Villani analysis rests is a precise quantitative estimate on the rate of entropy production for the spatially homogeneous Boltzmann equation. That is, consider a spatially uniform density $f(v,t)$. For such a density, Cercignani had conjectured that there should be some positive constant C depending only on the collision rate function B such that for all such solutions $f(v,t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(S[M] - S[f(\cdot,t)]) \leq -C(S[M] - S[f(\cdot,t)]) \ ,$$

where M is the global Maxwellian density with the same mean and variance as f. (Since energy and momentum are conserved, M does not depend on time.) This would yield the estimate

$$(S[M] - S[f(\cdot,t)]) \leq e^{-Ct}(S[M] - S[f(\cdot,0)]) \ ,$$

and then a well-known entropy inequality would imply that $f(\cdot,t)$ is converging to M in L^1 at the rate $e^{-tC/2}$.

The Cercignanni conjecture turns out to be false, but a particularly beautiful paper [3] of Villani obtains something almost as good: For each $\epsilon > 0$, there is a computable constant $C_\epsilon < \infty$ such that for all solutions f,

$$\frac{\mathrm{d}}{\mathrm{d}t}(S[M] - S[f(\cdot,t)]) \leq -C_\epsilon(f)(S[M] - S[f(\cdot,t)])^{1+\epsilon} \ . \tag{4}$$

The constant $C_\epsilon(f)$ depends on f, but only on properties of f that are well-behaved along the evolution. This result builds on a long chain of results, starting with the first one concerning the quantitative entropy production bounds [4, 5] by myself and M.C. Carvalho, through improvements [6] by Toscani and Villani, also yielding the bound (4), but with less precise information on how $C_\epsilon(f)$ depends on f, and then finally the result cited above.

Using the *full information* provided by Villani's result, in particular the precise dependence of $C_\epsilon(f)$ on ϵ and f, Desvillettes and Villani were able to show [1] that the entropy production rather quickly drives a solution of the spatially inhomogeneous Boltzmann equation toward a local Maxwellian. But then, the entropy production slows down dramatically – indeed it would stop completely if true local equilibrium were reached. The next part of the argument is to show that if the solution is not at a global equilibrium, and one waits a while, the effects of *streaming*, i.e., free motion between collisions eventually carry the solution away from the local Maxwellian, and entropy production then picks up again. The analysis, which is conditional on certain global regularity bounds, predicts the entropy will increase to its maximum along a smooth sort of staircase path, sometimes rapidly (when the solution is sufficiently far from local equilibrium) and sometimes with nearly no increase at all (when the solution is sufficiently close to local equilibrium).

Interestingly enough, this phenomenon, discovered in the course of proving the theorem, has since been observed in computer simulations. This is one indication of how close the mathematical technique is to the physics in this paper, as it is also in the work of Mouhot and Villani.

I shall close with a brief remark on Villani's style of analysis. As I have already indicated, it draws on close attention to the details of the underlying physical phenomena in the problems discussed here. But Villani is also a consummate analyst in the pure mathematical sense. Mathematical analysis can be described as the art of making precise quantitative estimates of interesting things by simple things. A great analyst has a great "nose" for finding the right "simple thing", which is not always suggested by the physics. In the early work [4, 5] of myself and Carvalho on the Cercignani conjecture, we compared the entropy production for physically interesting interactions with the entropy production for an especially simple variant of the Maxwellian molecules collision kernel – the one in which B is simply constant, independent of both $|v - v_*|$ and $\cos\theta$. For this simple collision kernel, one can compute a number of things, and then one compares. The result is an estimate providing a quantitative bound on the convergence rate, but not nearly as incisive as Villani's estimate.

Villani realized a *much better* basis of comparison was with the entropy production for *super hard sphere* collisions, which is the name for the purely mathematical model in which B is proportional to $|v-v_*|^2$. Recall that in the hard sphere case, it is proportional to $|v - v_*|$. There is no interaction between particles that leads to such a collision rate, but previous work providing counterexamples to Cercignani's conjecture worked all the way up to this limit on the power of $|v - v_*|$. Villani had the brilliant insight that not only did the counterexamples cease to work in the super-hard sphere case, *the Cercignani conjecture was actually true in this case*. He proved this, and then found a way to compare the entropy production for physically interesting collision rules, hard spheres say, to that of the super hard sphere, and in this way obtained [3] the very precise entropy production estimates for the spatially homogeneous Boltzmann equation that are one cornerstone of the work with Desvillettes on the spatially inhomogeneous Boltzmann equation.

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The 2010 Nobel prize in physics

by Alessandro Giuliani & Vieri Mastropietro (Rome, Italy)


Andre Geim and **Konstantin Novoselov**, both graduated from the Moscow Physics and Technology Institute (MFTI) and currently of the University of Manchester, have been awarded the 2010 Nobel prize in physics for the production, isolation and identification of **graphene**, a two-dimensional (2D) crystalline material composed of a single atomic layer of carbon. It is the first truly 2D crystalline material ever realized, and it has unique properties, which make it very interesting both for fundamental science and for future technological applications.

Indeed, the first surprise from such a material comes from its very existence since, according to several theoretical arguments (e.g., the Mermin-Wagner theorem), a 2D crystal should be thermodynamically unstable against thermal fluctuations breaking the perfect translational invariance or the perfect flatness of the sample. Among the several remarkable and counter-intuitive properties characterizing graphene, let us also mention its surprisingly high mechanical strength and rigidity, the presence of a minimum conductivity even in the absence of charge carriers, and the anomalous integer quantum Hall effect characterizing its behavior in an external magnetic field.

In view of these properties, one could perhaps think that it is very hard to find and isolate such an exotic material in nature, but it is not at all the case. In fact, a graphene layer is just one of the hexagonal carbon sheets that, stacked on the top of each other, form a three-dimensional crystal of graphite, the most common allotrope of carbon. Therefore, as Geim and Novoselov like to say, graphene is there also when you write with a pencil on a sheet of paper. The idea of Geim and Novoselov for isolating a single layer of graphite could not be simpler: just use adhesive tape to rip off thin flakes from graphite, and then repeat the operation again and again. The crucial point is to show that the result of this exfoliation procedure is really a one atom thick layer of carbon; Geim and Novoselov were originally able to show this by posing their sample on a plate of oxidized sylicium and by using an optical method.

Theoretically, electrons in graphene are described by a fermionic tight binding Hamiltonian on a 2D honeycomb lattice, whose analysis in the absence of interactions dates back to Wallace (1947). Later on, Semenoff (1984) observed that such system at low energies admits an effective description in terms of relativistic massless Dirac fermions; if an interaction between electrons is considered, one gets an effective description in terms of a sort of QED model in 2+1 dimensions. It was indeed well known since the time of Tomonaga (1950) that a non relativistic gas of fermions in one dimension admits an effective description in terms of relativistic Dirac fermions in 1+1 dimensions; the analogue is true also in 2D, provided that a honeycomb lattice is considered and the density is fixed exactly at half filling.

This "transmutation" of systems of massive non-relativistic electrons into relativistic

massless Dirac particles is typical of quantum low dimensional systems and is a phenomenon that attracted and fascinated theoreticians for decades.

The discovery of graphene now offers a real system in which such a phenomenon takes place. In particular, the fact that the effective description of the charge carriers in graphene is close to QED in 2+1 dimensions opens the possibility of observing the low-energy condensed matter analogues of several exotic phenomena that have first been predicted to happen in high energy experiments of particle physics, such as chiral symmetry breaking and spontaneous mass generation; namely, as Geim and Novoselov like to say, it opens the possibility of observing "QED in a pencil".

The discovery of graphene has a particular relevance for mathematical physics. Indeed it is a standard method of mathematical physics to consider models defined in lower dimensions, which are often easier to treat in a rigorous way than their more realistic counterparts in three dimensions; a celebrated example is the Ising model, which can be solved in two dimensions but not in three. Interestingly, graphene provides a new and remarkable physical realization of a low dimensional model, which adds to the list already including nanowires, 2D optical lattices, and micromagnets, just to mention a few; these systems prove that restricting to low dimensions is not just an academic game for theoreticians, but it has a strong physical relevance. In the case of graphene, this is particularly true, given the exciting technological perspectives that it opens. Moreover, as lower dimensions are usually more accessible to rigorous methods, the physics of graphene offers a number of theoretical problems, ranging from the quantum Hall effect to the role of Coulomb interactions and the possible emergence of spontaneous symmetry breaking, for which the methods of mathematical physics seem quite promising and well suited, even more than several standard approaches developed within the solid state physics community. Some progress has already been made in the mathematical comprehension of the low temperature behavior of this material, and we strongly believe that graphene is the right context to start understanding a number of outstanding open problems in quantum many body physics, such as the fractional quantum Hall effect, the localization/delocalization transition in the presence of disorder and the emergence of magnetic or superconducting long range order.

Andre Geim

Konstantin Novoselov

NESS: Non-equilibrium steady states in statistical mechanics

by David Ruelle (Bures sur Yvette, France & Rutgers, USA)

David Pierre Ruelle is one of the founder of contemporary Mathematical Physics. First he obtained the diploma of Candidat Ingénieur Civil at the Faculté Polytechnique of Mons (Belgium), then he studied physics at the Université Libre de Bruxelles, obtaining a Ph.D. degree in 1959 (prepared at ETH Zurich). For his outstanding contributions to Quantum Field theory, classical and quantum Statistical Mechanics, and Dynamical System theory he has been awarded several prizes in mathematics and physics, including the Henri Poincaré Prize of the IAMP (2006). In 1964, David Ruelle became Professor at the Institut des Hautes Études Scientifiques (IHES), in Bures-sur-Yvette, France. Since 2000, he is an Emeritus Professor at IHES and distinguished visiting professor at Rutgers University.

About ten years ago, David Ruelle started a serious mathematical study of non-equilibrium Statistical Mechanics from the point of view of differentiable dynamical systems. This approach is centered on the non-equilibrium steady states (NESS).

Statistical mechanics explains the bulk properties of matter in terms of the interactions of its microscopic constituents. In some situations, called *equilibrium*, one can define a temperature T, and the probability of encountering a system in a state of energy E is proportional to the Boltzmann factor $\exp(-E/kT)$ (where k defines the appropriate temperature scale). Equilibrium Statistical Mechanics permits the calculation of *specific heats*, and many other things. It has developed during the 20th century into a formidably efficient tool to understand certain properties of matter. But if you want to compute *heat conductivity*, it is of no help. This is because, to understand heat conductivity, and other characteristics of matter called *transport coefficients*, you need to know the microscopic time evolution (the latter is ignored by equilibrium Statistical Mechanics: it doesn't appear in Boltzmann factors).

Around 1950, non-equilibrium Statistical Mechanics started to be analyzed close to equilibrium by such people as Lars Onsager, Mel Green, and Ryogo Kubo. Their work (which led to the *Onsager Reciprocity Relations*, the *Fluctuation-Dissipation Theorem*, etc.) can be understood as first order perturbation theory around equilibrium. As usual, first order perturbation theory requires an expectation value of the perturbation to be computed in the unperturbed state. The unperturbed state is here the well understood equilibrium state; this leads to explicit formulas for the heat conductivity and other transport coefficients. The formulas in question involve the microscopic time correlations in the equilibrium state, and have constituted tremendous conceptual progress.

From a fundamental point of view, however, first order perturbation theory is, in the present case, an uncontrolled approximation. This was pointed out for instance by Nico van Kampen. The next task is thus to try to develop a rigorous theory of non-equilibrium away from equilibrium. The difficulty of this task is illustrated by the fact, repeatedly stressed by Joel Lebowitz, that we have no rigorous understanding of *Fourier's law*. (Fourier's law says that if the two ends of a long heat conductor are kept at slightly different temperatures, there will be a heat current proportional to the temperature difference and inversely proportional to the length of the conductor.)

To study a system kept away from equilibrium by some external forces, we have to couple the system to a *thermostat*: this will absorb the heat produced by the action of the forces. (Only if we stay infinitely close to equilibrium can the thermostat be ignored.) We want thus to find models of *thermostated* systems, which are physically realistic and nontrivial, yet mathematically accessible. That, as it turns out, is a tall order. Yet, a class of satisfactory models has emerged in recent years, and this is what I now want to describe. Let me however make clear that there are a number of interesting directions of research in non-equilibrium Statistical Mechanics. From now on I shall look at just one of them.

We shall be interested in classical systems with a finite number N of degrees of freedom. To a given Hamiltonian $H(\mathbf{p},\mathbf{q}) = \mathbf{p}^2/2m + V(\mathbf{q})$ there corresponds the evolution equation

$$\frac{d}{dt}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}-\partial_{\mathbf{q}}V\\ \mathbf{p}/m\end{pmatrix} ,$$

and the equilibrium state is $\sim \exp(-H(\mathbf{p},\mathbf{q})/kT)d\mathbf{p}\,d\mathbf{q}$ at temperature T. Let us replace $-\partial_{\mathbf{q}}V$ by a nongradient force $\xi(\mathbf{q})$ in order to produce non-equilibrium. Since energy is no longer conserved, the kinetic energy $\mathbf{p}^2/2m$ typically will tend to ∞ when $t \to \infty$: the system heats up. We need a thermostat. Bill Hoover and Denis Evans have had the idea of introducing *deterministic* thermostats, in particular the *isokinetic thermostat*: to the nongradient force ξ we add a thermostatic term $-\alpha\mathbf{p}$, where $\alpha = \xi\cdot\mathbf{p}/\mathbf{p}^2$. The resulting time evolution

$$\frac{d}{dt}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\xi-\alpha\mathbf{p}\\ \mathbf{p}/m\end{pmatrix} , \qquad (*)$$

keeps the kinetic energy $K = \mathbf{p}^2/2m$ constant. (This corresponds to a constant temperature $T = 2K/k(N-1)$.) We may thus restrict $(*)$ to the manifold $M = \{(\mathbf{p},\mathbf{q}) : \mathbf{p}^2/2m = K\}$, which will be assumed to be compact.

Dynamical systems of the special form $(*)$ are quite interesting, especially when ξ is locally a gradient. But we can also allow M to be a general compact manifold, replace $(*)$ by a (fairly) general time evolution, and establish a correspondence, or dictionary, relating physical concepts of non-equilibrium Statistical Mechanics, and mathematical concepts of differentiable dynamics on M. We now describe such a correspondence.

(1) *The deterministic time evolution (f^t) of a thermostatted non-equilibrium system on the phase space M is identified with a differentiable flow (f^t) on a compact manifold M.*

The flow is defined by

$$\frac{dx}{dt} = \mathcal{X}(x) \qquad , \qquad x(t) = f^t x(0) \ .$$

It is often possible and useful to assume *reversibility*, i.e., the existence of a time-reversal symmetry: a diffeomorphism i of M such that

$$i^2 = \text{identity} \qquad , \qquad i f^t i = f^{-t} \ .$$

Also, instead of using a continuous time, it is often mathematically convenient to deal with a discrete time t (i.e., f^t is the t-th iterate of a differentiable map $f : M \to M$).

(2) *Chaotic hypothesis (Giovanni Gallavotti - Eddie Cohen): (f^t) is assumed to be uniformly hyperbolic on a compact attractor $K \subset M$, containing no fixed point.*

This means that the tangent bundle $T_K M$ is a sum $E^s \oplus E^u \oplus E^0$ where E^0 is one-dimensional tangent to the direction of the flow, and for all $t \geq 0$,

$$||Tf^t \mathbf{v}|| \leq C\theta^t ||\mathbf{v}|| \quad \text{if} \quad \mathbf{v} \in E^s \qquad , \qquad ||Tf^{-t} \mathbf{v}|| \leq C\theta^t ||\mathbf{v}|| \quad \text{if} \quad \mathbf{v} \in E^u \ ,$$

where $C > 0, 0 < \theta < 1$.

Examples show that some simple time evolutions (for instance completely integrable) are unsuitable to describe non-equilibrium (for instance the behavior of heat conduction is pathological). One is thus led to assume some kind of chaotic time evolution. Uniform hyperbolicity is an attractive option because of the detailed mathematical knowledge we have of uniformly hyperbolic systems. This is a natural idea, and Giovanni Gallavotti credits me with having expressed it in a seminar talk, but to make the idea work, and obtain nontrivial results, was the achievement of Gallavotti and Cohen (see below: the Fluctuation Theorem). Here again it is convenient to consider the discrete time situation, for instance Anosov diffeomorphisms.

(3) *NESS = SRB measure ρ with support on attractor K.*

To study non-equilibrium we shall concentrate on the NESS (Non-equilibrium Steady State), which is the physically relevant ergodic measure for the time evolution (f^t). In general, no invariant measures are absolutely continuous on M, but the following are natural to consider:

$$\rho = \lim_{\tau \to \infty} (f^\tau)^* \text{ normalized Lebesgue (in the basin of } K \text{ in } M, \text{ in some sense}) \ ,$$

or

$$\rho = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \, \delta_{f^t x} \text{ (for } x \text{ in the basin of } K \text{ in } M, \text{ in some sense}) \ .$$

Such measures are loosely called SRB states. They have been analyzed in detail in the uniformly hyperbolic case by Yasha Sinai, David Ruelle, and Rufus Bowen (hence the name SRB). A general study without the uniform hyperbolicity assumption is due to François Ledrappier, Jean-Marie Strelcyn, and Lai-Sang Young. The discussion of SRB

measures in a number of specific situations constitutes an important chapter of contemporary dynamical systems theory.

At equilibrium the NESS should turn into the microcanonical ensemble of equilibrium Statistical Mechanics. For our purposes this means, loosely speaking, that $\rho(dx)$ is equivalent to the Lebesgue measure on M, or $\rho(dx) = dx$.

(4) *Entropy production rate = volume contraction rate in phase space.*

Starting from the Gibbs expression for the entropy associated with an absolutely continuous probability measure on M one obtains the entropy production rate $e(\rho)$ for the NESS ρ in the limit $\tau \to \infty$:

$$e(\rho) = \rho(-\mathrm{div}\mathcal{X}) \ .$$

Numerical simulations confirm this identification first proposed by Ladislav Andrey.

We shall now discuss a couple of consequences of (1)-(4).

(5) *The Gallavotti-Cohen Fluctuation Theorem.*

In the NESS ρ, the entropy production over time τ divided by $e(\rho)\tau$ fluctuates with a probability distribution which we denote by $P^\tau(\epsilon)d\epsilon$. The Fluctuation Formula is then

$$\lim_{\tau\to\infty} \frac{1}{\epsilon e(\rho)\tau} \log \frac{P^\tau(\epsilon)}{P^\tau(-\epsilon)} = 1 \ .$$

Note that this formula contains no adjustable parameter. It was discovered in numerical experiments by Denis Evans, Eddie Cohen, and Gary Morriss. Remarkably, the formula has a rigorous proof, due to Gallavotti and Cohen: it follows from the chaotic hypothesis and reversibility.

(6) *Linear response.*

Let us introduce a small time-dependent perturbation X_t in the time evolution equation, so that it becomes

$$\frac{dx}{dt} = \mathcal{X}(x) + X_t(x) \ .$$

Then the NESS ρ becomes $\rho + \delta_t\rho$, and a formal first order perturbation calculation gives

$$\int \delta_t\rho(dx)A(x) = \int_{-\infty}^{t} d\tau \int \rho(dy)X_\tau(y)\cdot\partial_y A(f^{t-\tau}y)$$

(This is the so-called *linear response* to the perturbation.) Physicists like to discuss the case when X_t is a periodic function of t, of frequency ω, and are thus interested in the *susceptibility* function

$$\omega \mapsto \int_0^\infty e^{i\omega t}dt \int \rho(dx)\, X(x)\cdot\partial_x(A\circ f^t)$$

This is a one-sided Fourier transform, and therefore expected to be analytic in the upper complex ω half-plane. In traditional treatments the one-sidedness is associated with

causality (the effect of a perturbation cannot precede the perturbation) and the upper half-plane analyticity leads to *dispersion relations*: the real and imaginary parts of the susceptibility are related by Hilbert transform. (Dispersion relations also appear in the theory of quantum scattering, and have played an important role in high energy physics.)

The traditional treatments just referred to were close to equilibrium. In our presentation, this means that $\rho(dx)$ is equivalent to Lebesgue on M ($\rho(dx) = dx$), and an integration by parts gives

$$\int \rho(dx)\, X_\tau(y) \cdot \partial_y (A \circ f^t) = - \int dx\, (\mathrm{div}_x X) A(f^t x) \ ,$$

i.e., the susceptibility is the one-sided Fourier transform of a time correlation function in the equilibrium state. The susceptibility, which is a non-equilibrium ("dissipative") quantity is thus related to time correlations in equilibrium ("fluctuations"). We recover thus a form of the *Fluctuation-Dissipation Theorem*.

The formal discussion close to equilibrium just presented can be replaced by a rigorous analysis away from equilibrium under suitable assumptions on the time evolution. In the case of Anosov diffeomorphisms, the basic result of differentiability of $f \mapsto \rho$ has been established by Anatole Katok, Gerhard Knieper, Mark Pollicott, and Howie Weiss. The connection with the susceptibility and the analyticity of the latter were later obtained by Ruelle for uniformly hyperbolic diffeomorphisms. The case of uniformly hyperbolic flows was then handled by Ruelle, and with different tools by Oliver Butterley and Carlangelo Liverani.

In the uniformly hyperbolic case the dispersion relations are thus proved away from equilibrium, which is a new and very interesting result from the physical viewpoint (physical applications have been made by Valerio Lucarini). Also, corresponding to the decomposition $X = X^s + X^{u0}$, with $X^s \in E^s, X^{u0} \in E^u + E^0$ we may write the susceptibility as

$$\int_0^\infty e^{i\omega t} dt \int \rho(dx)\, X^s(x) \cdot \partial_x (A \circ f^t) - \int_0^\infty e^{i\omega t} dt \int \rho(dx)\, (\mathrm{div}_x^{u0} X^{u0}) A(f^t x) \ .$$

Omitting mathematical details, let me explain the basic physical meaning of this formula. The first term corresponds to "relaxation to the attractor". The second term is the one-sided Fourier transform of a correlation function, and is what remains of the Fluctuation-Dissipation Theorem when we are away from equilibrium.

An optimistic view would be that the above results, proved for uniformly hyperbolic dynamics (i.e., under the chaotic hypothesis) are quite general. This is not true: the analyticity of the susceptibility in the upper half-plane is violated in some non-uniformly hyperbolic examples. This apparent "violation of causality" is due to the fact that $t \mapsto \int \rho(dx)\, X(x) \cdot \partial_x (A \circ f^t)$ can grow exponentially: our non-equilibrium system can give energy to the outside world, so that a small perturbation is amplified exponentially.

An obvious problem, at this time, is to escape the straitjacket of uniform hyperbolicity, and find a large class of systems with reasonable behavior from the point of view of non-equilibrium Statistical Mechanics. There are some encouraging results in this direction (by Viviane Baladi, Dima Dolgopyat, the Brazilians (Marcelo Viana, Artur Àvila , ...), ..., and also myself), but we are trying to find a way into the impenetrable jungle of general dynamical systems, and this may be formidably difficult.

I have written two review articles on the theme of the present introductory note, giving more results, and background literature[1]:

Smooth dynamics and new theoretical ideas in non-equilibrium Statistical Mechanics [J.Stat.Phys. **95**,393-468(1999)],

A review of linear response theory for general differentiable dynamical systems [Nonlinearity **22**,885-870(2009)].

[1]These may be accessed from my homepage www.ihes.fr/~ruelle/Publications as items [124], [151].

Much ado about something: why Lieb-Robinson bounds are useful

by Bruno Nachtergaele & Robert Sims (Davis & Tucson, USA)


Professor Bruno Nachtergaele (Department of Mathematics, University of California, Davis) and Professor Robert Sims (Department of Mathematics, University of Arizona) are well-known researchers in Equilibrium and Non-Equilibrium Statistical Mechanics. Bruno Nachtergaele's primary interest for many years has been quantum spin systems. Robert Sims established his reputation with important contributions in random lattice models, such as the Anderson Hamiltonian and the Sherrington-Kirkpatrick model. Their recent work on Lieb-Robinson bounds, which elucidate the locality structure of non-relativistic quantum dynamics, is having a significant impact on our understanding of complex states of quantum many-body systems relevant for Condensed Matter Physics, Materials Science, and Quantum Information Theory.

Understanding many-body dynamics lies at the heart of many fundamental problems of mathematical physics. Even when one is not directly concerned with time-dependent phenomena, such as in the study of equilibrium and non-equilibrium stationary states, one is essentially investigating properties of the dynamics. It is also possible to approach spectral questions about the Hamiltonian of a quantum system starting from an analysis of the dynamics it generates.

The equations of hydrodynamics, the Boltzmann equation, and the Gross-Pitaevskii equation, are well-known examples that continue to receive a lot of attention. In these examples the aim is to describe the dynamics of a large number, N, of identical particles. More recently, other types of questions have been raised in quantum information theory, where one is less interested in the large N limit (although N may be large) and where one usually does not want to make the assumption that all degrees of freedom are identical and are subject to identical interactions. In either case, one is confronted with the task of taming the complexity of many-body dynamics, and quantum mechanics adds another layer of complexity due to the role played by entanglement. Fortunately, the interactions in many physical systems are either of finite range or their strength decays exponentially (or with a large inverse power) in the distance between particles. In the past five years Lieb-Robinson bounds have been shown to be a powerful tool to turn this inherent locality of physical systems into useful mathematical estimates. Lieb-Robinson bounds provide an estimate for the speed of propagation of signals (disturbances) in a spatially extended system and estimate the magnitude of signals propagating faster than this speed (the propagation bound is not absolute since we are dealing with non-relativistic systems).

Systems defined on a lattice or, more generally, a metric graph, are simpler than systems of particles in the continuum because the number of degrees of freedom in a finite region of space can be bounded. So far, Lieb-Robinson bounds have only been obtained

for such systems but we believe that similar results are possible for continuum systems and would also be very useful in applications. We will only consider lattice systems here, with the term "lattice" taken in the broad sense of a discrete set of points. For concreteness and due to space-limitations, we will restrict ourselves to quantum systems. Similar considerations have been applied to classical systems such as anharmonic lattice oscillators [23, 9, 37] but we will not discuss this further here. Our basic set-up is then as follows.

Let Γ be a set equipped with a metric d. Associated to each $x \in \Gamma$, there is a Hilbert space $\mathcal{H}_x$ and a self-adjoint operator H_x defined on a dense domain $\mathcal{D}_x \subset \mathcal{H}_x$. For example, we could have a harmonic oscillator at each site x. In other applications, consider the isotropic nearest-neighbor Heisenberg model in the absence of a magnetic field, we may have $H_x = 0$. The dynamics is then entirely due to the interactions, which we will introduce in a moment. Thus, we have a multi-component quantum system defined on Γ, and for $x, y \in \Gamma$, $d(x, y)$ is interpreted as the distance between the subsystems located at x and y. We allow for the possibility that Γ is infinite, e.g., $\Gamma = \mathbb{Z}^\nu$. For any finite $\Lambda \subset \Gamma$, the Hilbert space of the associated subsystem $\mathcal{H}_\Lambda$ and the corresponding algebra of observables $\mathcal{A}_\Lambda$ are given by $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\mathcal{A}_\Lambda = \bigotimes \mathcal{B}(\mathcal{H}_x)$, where $\mathcal{B}(\mathcal{H}_x)$ denotes the bounded linear operators over $\mathcal{H}_x$. For an increasing sequence of finite subsets $\Lambda_n \uparrow \Gamma$, the algebra of local observables is given by the inductive limit $\mathcal{A}_{\rm loc} = \cup_n \mathcal{A}_{\Lambda_n}$. This makes sense because $\mathcal{A}_{\Lambda_n}$ is naturally embedded in $\mathcal{A}_{\Lambda_m}$, for all $m > n$ by identifying A and $A \otimes \mathbb{1}$, where $\mathbb{1}$ is the identity operator on $\mathcal{H}_{\Lambda_m \setminus \Lambda_n}$. With this identification, for all $A \in \mathcal{A}_{\rm loc}$ we can define the support of A, as the smallest Λ such that A belongs to the subalgebra $\mathcal{A}_\Lambda$. The local Hamiltonians for such a system are defined in terms of an interaction. An interaction is a mapping Φ from the set of finite subsets of Γ into $\mathcal{A}_{\rm loc}$ such that for each finite $X \subset \Gamma$, $\Phi(X)^* = \Phi(X) \in \mathcal{A}_X$. Then, a family of local Hamiltonians H^Φ_Λ, $\Lambda \subset \Gamma$ finite, is defined by

$$H^\Phi_\Lambda = \sum_{x\in\Lambda} H_x + \sum_{X\subset\Lambda} \Phi(X)\,. \tag{5}$$

When Φ is understood, we often suppress it in our notation. Since the sum in (5) above is finite, and each $\Phi(X)$ is bounded, H_Λ is self-adjoint on $\mathcal{H}_\Lambda$, and therefore the Heisenberg dynamics

$$\tau^\Lambda_t(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for all } A \in \mathcal{A}_\Lambda\,, \tag{6}$$

is well-defined.

The Lieb-Robinson bounds depend on a combination of properties of Γ and Φ. If Γ is infinite, it is necessary to impose a condition, roughly equivalent to finite-dimensionality, as follows. We assume that there is a non-increasing, real-valued function $F : [0, \infty) \to (0, \infty)$, with two properties:
i) *uniform integrablility:*

$$\|F\| = \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x, y)) \,<\, \infty\,, \tag{7}$$

ii) *convolution property:* there exists a number $C > 0$ such that for any pair $x, y \in \Gamma$,

$$\sum_{z\in\Gamma} F(d(x,z))F(d(z,y) \leq CF(d(x,y)) \ . \tag{8}$$

For the case of $\Gamma = \mathbb{Z}^\nu$, one choice of F is given by $F(r) = (1+r)^{\nu+1}$. Then the convolution property holds with $C = 2^{\nu+1}\sum_{x\in\Gamma} F(|x|)$. Note that one can assume $C = 1$ without loss of generality (replace F by $C^{-1}F$.) An important observation is that if there exists a function F on Γ satisfying i) and ii), then for any $\mu \geq 0$, the function F_μ defined by setting $F_\mu(r) = e^{-\mu r}F(r)$ also satisfies i) and ii) with $\|F_\mu\| \leq \|F\|$ and $C_\mu \leq C$. For any $\mu \geq 0$, we denote by $\mathcal{B}_\mu(\Gamma)$ the set of interactions Φ for which

$$\|\Phi\|_\mu = \sup_{x,y\in\Gamma} \frac{1}{F_\mu(d(x,y))} \sum_{\substack{X\subset\Gamma:\\ x,y\in X}} \|\Phi(X)\| \ < \infty \ . \tag{9}$$

If $\Phi \in \mathcal{B}_\mu(\Gamma)$, then a Lieb-Robinson bound of the form

$$\left\| \left[\tau_t^\Lambda(A), B\right] \right\| \leq 2\|A\|\|B\|C_\mu^{-1}\left(e^{2C_\mu\|\Phi\|_\mu|t|} - 1\right) \sum_{x\in X}\sum_{y\in Y} F_\mu(d(x,y)) \tag{10}$$

holds for all $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $X \cap Y = \emptyset$, and $t \in \mathbb{R}$. If $\mu > 0$, the double sum can be bounded by an exponentially decaying factor of the form $C\|F\|e^{-\mu d(X,Y)}$, which leads to a version of the bound in the familiar form:

$$\left\| \left[\tau_t^\Lambda(A), B\right] \right\| \leq 2\|A\|\|B\|C \ e^{-\mu(d(X,Y)-v|t|)} \ . \tag{11}$$

Here, $v = 2\mu^{-1}C_\mu\|\Phi\|_\mu$ is the Lieb-Robinson velocity and one can take $C = C_\mu^{-1}\min(|X|,|Y|)$ or in the case of interactions of finite range R, $C = RC_\mu^{-1}\min(|\partial X|, |\partial Y|)$, where $|\partial Z|$ denotes the size of the boundary of Z. A brief discussion of the literature is in order here. The original bound of the form (11), albeit in a slightly more restricted setup, appeared in 1972 in [20] (see also [4]). In the following three decades, apart from a few works considering classical lattice oscillators [23] and calculations for specific models [36], applications and extensions of Lieb-Robinson bounds received little attention. This changed in 2004 with Hastings' work [13] on the multi-dimensional Lieb-Schultz-Mattis Theorem [21]. In his paper, Hastings used Lieb-Robinson bounds in combination with his "quasi-adiabatic continuation" technique to analyze properties of ground states of quantum lattice systems and their excitations. For a self-contained presentation of quasi-adiabatic continuation see [2]. This technique and Lieb-Robinson bounds were subsequently extended and used in new applications by Hastings and collaborators [18, 16, 5, 14, 17, 15, 7, 6] and other authors [28, 25, 29, 8, 32, 10, 11, 30, 26, 12, 34, 37, 27, 31, 35, 33].

The new applications motivated extensions of the Lieb-Robinson bounds to more general systems as well as improvements on the basic estimates. We will mention some of these newer results below but space limitations will not allow us to explicitly mention

all the results that have appeared in the past few years. First, however, we clarify the relation between bounds on commutators and the support of observables.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose there is an $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that for all $B \in \mathcal{B}(\mathcal{H}_2)$ we have $\|[A, \mathbb{1} \otimes B]\| \leq \epsilon \|B\|$. Then, there exists $A_\epsilon \in \mathcal{B}(\mathcal{H}_1)$ such that $\|A - A_\epsilon\| \leq \epsilon$. In fact, if $\mathcal{H}_2$ is finite-dimensional A_ϵ can be obtained using the partial trace of A over $\mathcal{H}_2$, but the result also holds for bounded observables on infinite-dimensional Hilbert spaces [2]. For fixed $A \in \mathcal{A}_X$, one can apply this result with $\tau_t^\Lambda(A)$ taking the role of A. By letting $\mathcal{H}_1 = \mathcal{H}_{X_r}$, where $X_r = \{y \in \Lambda, d(X, y) \leq v|t| + r\}$ and $\mathcal{H}_2 = \mathcal{H}_{\Lambda \setminus X_r}$, we see that (11) gives a bound for the error one makes if one replaces $\tau_t^\Lambda(A)$ by its best approximation with support contained in X_r. Therefore, it is not surprising that Lieb-Robinson bounds can be used to prove the existence of the dynamics in the thermodynamic limit [4]. While Lieb-Robinson bounds do not always give the best possible estimates, in some cases they do, and moreover, they have been used to prove the only known results for anharmonic lattices [27, 1].

Another fundamental result that directly relies on the locality properties expressed by Lieb-Robinson bounds is the Exponential Clustering Theorem. It states that Hamiltonians with a non-vanishing spectral gap have ground states with exponentially decaying spatial correlations. Although the corresponding result in relativistic quantum field theory has been known for a long time, the non-relativistic version, with the Lieb-Robinson velocity playing the role of the speed of light, was only proved in 2006 [28, 16]. The correlation length ξ satisfies $\xi \leq 2v/\gamma + 1/\mu$, where v is the Lieb-Robinson velocity, γ is the spectral gap above the ground state, and μ is the parameter measuring the exponential rate of decay of the interaction. When one uses a more recent version of the Lieb-Robinson bounds [30], the exponential clustering for gapped ground states ω of one-dimensional systems with short-range interactions, can be shown to hold in the following strong form:

$$|\omega(AB) - \omega(A)\omega(B)| \leq c\|A\| \, \|B\| \exp\left(-d(X, Y)/\xi\right) ,$$

where X and Y are the supports of A and B, respectively. Note that only the distance between the supports and not their size appears in the estimate, a feature that was exploited by Matsui in his investigation of the split property for quantum spin chains [24].

The Area Law, the conjecture that the entropy of the restrictions of gapped ground states of quantum lattice models to a finite volume Λ grows no faster than a quantity proportional to the surface area of Λ, has been proved by Hastings for one-dimensional systems in [14]. In his paper Lieb-Robinson bounds are used to derive an approximate factorization property of the density matrices of gapped ground states. Such a result can be generalized to higher dimensions [12], but it does not, by itself, suffice to prove the Area Law in this context. This issue remains a topic of active investigation (see, e.g., [3]).

The estimate of the correlation length in terms of the gap plays an important role in several other applications of Lieb-Robinson bounds, including the multi-dimensional Lieb-Schultz-Mattis (LSM) theorem. A precise statement of the general multi-dimensional LSM theorem would be too long to fit in the space alotted, but since it was one of

the first non-trivial applications of Lieb-Robinson bounds, it deserves to be discussed here. For concreteness consider the spin-S nearest-neighbor isotropic quantum Heisenberg antiferromagnet with $S = 1/2, 3/2, 5/2, \ldots$, defined on a finite subset of $\mathbb{Z}^\nu$ of the form $[1, 2L] \times [1, 2L+1]^{\nu-1}$, and with periodic boundary conditions in the first coordinate. It is known that this model has a unique ground state [22]. The LSM theorem [13, 29] then provides a bound for the energy of the first excited state:

$$E_1 - E_0 \leq C \, \frac{\log L}{L} \ ,$$

where C is a constant of order 1, only depending on the dimension and the coupling constant. The proof is by showing that a gap *larger* than the bound claimed above would allow one to construct variational states of *lower* energy, which is a contradiction. The properties of the variational states, the estimate of their energy and the proof of orthogonality to the ground state, rely on Lieb-Robinson bounds in an essential way. See [30, Section 5] for a more detailed outline of the complete proof. Here we just mention that Lieb-Robinson bounds allow one to show that, as long as the spectral gap is not too small, local perturbations added to a Hamiltonian modify the ground state only in a neighborhood of the perturbation. See [2] for a general proof of this property.

In an impressive application of the adiabatic continuation technique and Lieb-Robinson bounds Hastings and Michalakis proved the quantization of the Hall conductance for a general class of models of interacting fermions on a lattice in [17]. An extension to the fractional quantum Hall effect is also discussed in this work.

Bravyi, Hastings, and Michalakis [7, 6] have shown that topological order in the ground states of a class of Hamiltonians that are the sum of commuting short-range terms, such as Kitaev's toric code model [19], is stable under arbitrary sufficiently small short-range perturbations. This can be regarded as another instance where Lieb-Robinson bounds are used to show that local perturbations have only local effects, and therefore cannot destroy a global property such as topological order.

In this short review we have only considered Hamiltonian quantum dynamics. We just note that Lieb-Robinson bounds have also been derived for irreversible dynamics described by semigroups with a generator of Lindblad form [33].

The complexity of quantum dynamics and its tendency to create entangled states are a barrier to our intuitive understanding of many of the most interesting physical phenomena. Good mathematical results that elucidate the structure behind this complexity are essential to aid our understanding. We hope to have convinced the reader that Lieb-Robinson bounds are a good example of this kind of mathematical result.

Acknowledgment

Work supported by National Science Foundation grants DMS-0757581, DMS-1009502, and DMS-0757424. Article written during a visit of BN to the Institut Mittag-Leffler (Djursholm, Sweden).

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CIME school on quantum many body systems

The **CIME school** on **Quantum Many Body Systems** took place in Cetraro (Italy) from August 30 to September 4, 2010, in the beautiful setting of the Calabria seacoast. The school had 30 participants, coming from several different countries, including Austria, Denmark, France, Germany, Italy, UK, Ukraine and USA. Although the school was primarily intended for graduate students, the high level of the lectures also attracted several more senior researchers.

The hotel where the CIME school took place

The school consisted of four series of lectures, given by V. Rivasseau, R. Seiringer, J. P. Solovej and T. Spencer, and divided in four lessons of two hours each (two in the morning and one in the afternoon). In addition, one afternoon was devoted to short presentations of the research activity of the younger participants, and one evening to a short description of the activities of some of the senior participants.

The lectures of Prof. **Rivasseau** gave an introduction to some results in solid state physics obtained via the constructive Renormalization Group methods, with focus on the proof of Fermi liquid behavior for a system of non-relativistic two-dimensional fermions above the BCS transition temperature; one lecture was devoted to the exciting perspectives opened by the use of the same methods to the quantum gravity problem. The lectures of Prof. **Seiringer** gave an introduction to the problem of Bose-Einstein condensation and to the methods leading to its rigorous justification in the so-called Gross-Pitaevski limit, starting from the basic notions up to very advanced topics like the analysis of rotating traps and the emergence of lattices of quantized vortices. Prof. **Solovej** provided a careful introduction to the problem of the quantum Coulomb gas; he gave a self-consistent introduction to the functional analytic methods used to prove thermodynamic stability, following a recent improved proof that allows one to treat on the same footing

translation and non-translation invariant systems of charged fermions and bosons. Finally Prof. **Spencer** described the rigorous and powerful methods of supersymmetry and their application to the problem of the localization-delocalization transition in the Anderson model and in random matrices; moreover, he gave a tutorial review of some classical results and techniques, such as the use of Ward Identities in the XY model. The lecturers gave a draft of their notes to the students, now available at http://www.cime.unifi.it/ and at http://www.mat.uniroma3.it/users/giuliani/public_html/cime/, and a book with their lectures will be soon published by Springer, in the special series dedicated to the CIME summer schools.

The atmosphere at the school was wery stimulating, many questions and comments arose everyday at the end of each lessons and scientific discussions took place all through the day, both in the conference room and in the seaside terrace where lunch was offered daily. The students profited very much from the close and frequent interactions with the lecturers. The overall experience was very interesting and enjoyable, as confirmed by several extremely positive comments by the participants and the lecturers. We hope very much to be able to repeat this event in the future, with the help of the CIME foundation and of the International Association in Mathematical Physics.

The organizers
Alessandro Giuliani
Vieri Mastropietro
Jakob Yngvason

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. Benerji Babu Avula, Department of Mathematics and Humanities, National University of Technology, Warangal, India
2. Jens Bolte, Department of Mathematics, Royal Holloway, University of London, Egham, UK
3. Robert Carlson, Department of Mathematics, University of Colorado, Colorado Springs, Colorado, USA
4. Sven Gnutzmann, School of Mathematical Sciences, University of Nottingham, Nottingham, UK
5. Daniel Grieser, Institut für Mathematik, Carl von Ossietzky Universität, Oldenburg, Germany
6. Stephen Gustafson, Department of Mathematics, University of British Columbia, Vancouver, Canada
7. Jonathan Harrison, Department of Mathematics, Baylor University, Waco, Texas, USA
8. Paul Lammert, Dept. of Physics Pennsylvania State University, University Park, Pennsylvania, USA
9. Daniel Lenz, Mathematisches Institut Friedrich-Schiller-Universität, Jena, Germany
10. Shu Nakamura, Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, Japan
11. Leonid Parnovski, Department of Mathematics, University College London, London, UK
12. H.P. Rani Department of Mathematics National Institute of Technology, Warangal India
13. Tulkin Rasulov, Department of Algebra and Analysis, Bukhara State University, Bukhara, Uzbekistan
14. Tigran Sedrakyan, Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland, USA
15. Yutaka Shikano, Department of Physics, Tokyo Institute of Technology, Tokyo, Japan

16. Toshikazu Sunada, Department of Mathematics, Meiji University, Kawasaki, Japan
17. Christiane Tretter, Mathematical Institute University of Bern, Bern, Switzerland
18. Boris Vainberg, Department of Mathematics and Statistics, University of North Carolina, Charlotte, North Carolina, USA
19. Ivan Veselic, Fakultaet für Mathematik, Technische Universität Chemnitz, Chemnitz, Germany
20. Rudi Weikard, Department of Mathematics University of Alabama, Birmingham, Alabama, USA
21. Monika Winklmeier, Departamento de Matematicas, Universidad de los Andes, Bogota, Colombia
22. Enrique Zuazua, Basque Center for Applied Mathematics, Derio, Basque Country, Spain.

Open positions

- Deadline Nov 30, 2010: Tenure track Assistant Professor
(LINK: http://tiptop.iop.org/index.cfm?action=job.desc&jobid=14547)
Physics Department, Pontificia Universidad Catolica de Chile

Jan Philip Solovej (IAMP Secretary)