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Cover picture: Krysztof Gawędzki (1947-2021). Photo courtesy of Ignacy Gawędzki.

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Contents

Krzysztof Gawędzki – a master of quantum field theory

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Krzysztof Gawędzki was one of the giants of mathematical physics of the past 50 years. Born in 1947 at Zarki, Poland, he did his studies at the University of Warsaw where he got his PhD in 1971 and then continued during 1971-1981 as a researcher at the Department of Mathematical Methods in Physics in Warsaw. This institute led by Krzysztof Maurin gathered at that time some of the leading mathematical physicists in Poland and among them Krzysztof was particularly close to Tadeusz Bałaban, Jerzy Kijowski and Stanislav Voronovich.

His work spanned an astonishing range of topics including geometric quantisation, supermanifolds, renormalisation group, conformal field theory, turbulence, KAM theory, stochastic thermodynamics, topological insulators and even an experiment. Despite this variety his approach to research was not superficial; quite the contrary when embarking seriously on a new subject he worked years to reach a deep understanding of it. He was equally comfortable with rigorous proofs as with long and complicated calculations. A unifying theme of all his work is quantum field theory and he was one of the leading field theorists of the past five decades. I had the privilege to collaborate with him for more than twenty years and for that collaboration we shared the American Physical Society's 2022 Dannie Heineman Prize in mathematical physics with the citation "For fundamental contributions to quantum field theory, statistical mechanics, and fluid dynamics using geometric, probabilistic, and renormalization group ideas." In this article I will review some of his and our joint work on these subjects.

1 Renormalization Group

With his advisor Krzysztof Maurin, Krzysztof got a solid education in geometry and analysis which is visible in his later career. His thesis "On the geometrisation of the canonical formalism in classical field theory" is on the geometrical side, whereas his habilitation in 1976 "Fourier-like kernels in geometric quantisation" combines geometry and analysis. In the mid-70's Krzysztof decided he wanted to enter the rapidly developing field of constructive quantum field theory. This was not an easy task in those days for somebody working alone in Poland far away from where the action was. Glimm, Jaffe and Spencer had recently [44] proven the $\mathbb{Z}_2$ symmetry-breaking phase transition in ϕ_2^4 QFT and there was some debate in the community whether the existence of three low temperature phases could be established in the ϕ_2^6 QFT. Krzysztof showed in [27] that this can be done and in the process learned the state of the art of cluster and polymer expansion methods in Euclidean QFT. This exercise also convinced him that to go beyond the super-renormalisable cases of two and three dimensional QFTs towards construction of the physically relevant four dimensional ones new ideas are needed and these new ideas have to come from physics where the renormalisation group (RG) had transformed the understanding of QFT and 2nd order phase transitions. RG had been introduced to mathematical physicists by Blecher and Sinai [9] and Collet and Eckmann [17] in the context of Dyson's hierarchical model [19] and by Gallavotti's group in Rome [26, 5, 6]. Especially the latter served as an inspiration to our joint work that started during our stay at Arthur Jaffe's group at Harvard 1979-80.

The simplest interacting QFT model that had been studied in constructive field theory is the ϕ_d^4 model, a theory of a scalar field $\phi : \mathbb{R}^d \to \mathbb{R}$ with classical action functional

$$S_{m,\lambda}(\phi) = \int ((\nabla\phi)^2 + m^2\phi^2 + \lambda\phi^4)dx. \tag{1}$$

In Euclidean QFT one attempts to construct a probability law on some space $\mathcal{F}$ of fields ϕ formally given by the expectation on suitable observables $F : \mathcal{F} \to \mathbb{R}$

$$\langle F\rangle = Z^{-1} \int_{\mathcal{F}} F(\phi)e^{-S(\phi)} D\phi. \tag{2}$$

A natural approach is to perturb around the non-interacting case $\lambda = 0$ in which case the field is Gaussian, a random distribution with covariance given by the Green function

$$\langle\phi(x)\phi(y)\rangle = (-\Delta + m^2)^{-1}(x, y).$$

Due to the divergence of this expression as $x \to y$ this field is almost surely not a function but rather a generalised function with regularity $H^{-s}(\mathbb{R}^d)$, $s > \frac{d-2}{2}$. This singular behaviour makes the definition of (2) as a perturbation of the $\lambda = 0$ case problematic since one needs to make sense of $\phi(x)^4$ for a distribution. And indeed, a formal perturbation theory for (2) in powers of λ leads to divergent expressions ("ultraviolet problem"). Likewise, for $m^2 = 0$ the field is strongly correlated at long distances

$$\langle\phi(x)\phi(y)\rangle \sim |x - y|^{2-d}$$

for $d > 2$ leading to "infrared divergencies" in perturbation theory.

A scaling argument reveals the role the dimension d plays in the problem. The case of $m^2 = 0$ is the Gaussian free field which is scale invariant $\phi \overset{law}{=} \phi_\ell$ where $\phi_\ell(x) = \ell^{\frac{d-2}{2}}\phi(\ell x)$[1]. Substitution to (1) gives

$$S_{m,\lambda}(\phi) = S_{\ell m,\ell^{4-d}\lambda}(\phi_\ell) \tag{3}$$

so we are led to expect that at small spatial scales i.e. $\ell \to 0$ the parameter λ is irrelevant if $d < 4$ and at large spatial scales i.e. $\ell \to \infty$ this happens if $d > 4$. Thus the UV behaviour should be close to Gaussian in the former case and the IR behaviour at the critical point likewise in the latter case. It turns out that the exact scaling (3) does not hold for the probability law (2) due to renormalisation needed to define it but the above conclusions hold nevertheless.

Constructive QFT had solved the UV problem in $d < 4$ by defining the measure (2) as a limit as $\epsilon \to 0$ of regularised measures involving a small scale cutoff ϵ e.g. by defining the field on the lattice $\epsilon\mathbb{Z}^d$ instead of $\mathbb{R}^d$. Then the limit exists provided one adds an ϵ dependent term $\delta m^2(\epsilon)\phi^2$ to (1) where $\delta m^2(\epsilon) = a\lambda \log\epsilon$ in $d = 2$ and $\delta m^2(\epsilon) = b\lambda/\epsilon + c\lambda^2 \log\epsilon$ in $d = 3$ with explicit coefficients a, b, c. However, this approach fails in $d = 4$ where formally an infinite number of such diverging constants ("counterterms") is needed and also for the quartic coupling λ.

[1] Some care is needed in $d = 2$ where the field is defined only modulo constants

The Kadanoff-Wilson RG approach to QFT and critical phenomena is to view the problem via *effective actions* S_ℓ describing physics at spatial scale ℓ. S_ℓ has UV cutoff ℓ, e.g. in the form of lattice of spacing ℓ and $e^{-S_\ell(\phi)}D\phi$ (normalised to a probability measure) is the probability distribution of the field ϕ coarse grained to scale ℓ. The Renormalisation Group flow is the map

$$\ell \to S_\ell$$

and it describes how physics changes with scale. The UV problem is then to find a S_ϵ of the form (1) with ϵ dependent parameters such that starting with S_ϵ the resulting S_ℓ has a limit as $\epsilon \to 0$ for all $\ell > 0$. For the IR problem the ϵ is fixed, say to $\epsilon = 1$, and one inquires about the behaviour of S_ℓ as $\ell \to \infty$.

It is convenient (and necessary if one works on the lattice) to study RG flow in discrete steps $S_\ell \to S_{L\ell}$ where $L > 1$ is fixed. Then one has

$$e^{-S_{L\ell}(\phi)} = \int e^{-S_\ell(\phi+\zeta)}D\zeta \tag{4}$$

where ζ is a field on scales $[\ell, L\ell]$. Going to dimensionless variables (φ_ℓ above) this path integral has UV cutoff 1 and IR cutoff L and thus one ends up studying the iteration of a fixed map $\mathcal{R}$ acting on unit cutoff action functionals $H_\ell(\varphi) = S_\ell(\varphi_{\ell^{-1}})$ which can be thought as Hamiltonians of a classical spin system on $\mathbb{Z}^d$ where the spins $\varphi(x) \in \mathbb{R}$. This suggests that one could hope for a rigorous control of the UV problem if e^{-H_ℓ} becomes Gaussian as $\ell \to 0$ and of the IR-problem if this happens as $\ell \to \infty$ i.e.in the **asymptotically free** cases. The above scaling argument indicates this is the case for the UV problem in $d < 4$ and the IR problem for $d > 4$. The borderline case $d = 4$ requires a more detailed analysis of the RG flow and understanding such cases rigorously was the problem that Krzysztof and I set out to solve in 1979.

The analysis boils down to setting up the RG transformation in a way that can be be iterated. Since we are considering Hamiltonians H that are perturbations of a quadratic one we write

$$e^{-H(\varphi)}D\varphi = e^{-V(\varphi)}d\mu(\varphi) \ ,$$

where μ is a Gaussian measure preserved by the RG, i.e. its fixed point. The RG map $V \to RV$ consists of two steps: a coarse graining and a rescaling. Coarse graining is realised for the Gaussian measure by decomposing φ into a sum of independent fields

$$\varphi = \varphi' + \zeta$$

where φ' is the large scale part involving spatial scales $\geq L$ and ζ the small scale part involving scales in the range $[1, L]$. The covariance G of φ then decomposes as $G = G' + \Gamma$ and the coarse graining map becomes

$$e^{-V'(\varphi')} = \int e^{-V(\varphi'+\zeta)}d\mu_\Gamma(\zeta). \tag{5}$$

Finally, the rescaling is defined by $\varphi \to \varphi_{L^{-1}}$ so that $RV(\varphi) = V'(\varphi_{L^{-1}})$.

Let us apply the coarse graining map (5) to a local potential V of the form

$$V(\varphi) = \sum_{\mathbf{x}\in\Lambda} [(\mu\varphi(\mathbf{x})^2 + g\varphi(\mathbf{x})^4] \tag{6}$$

with $\Lambda \subset \mathbb{Z}^d$. The covariance Γ has exponential decay with decay rate $\mathcal{O}(L^{-1})$. Hence (5) is a problem of classical statistical mechanics in the high temperature regime. Furthermore if we start with μ and g small we expect the problem to be perturbative. We can formally expand the exponential in Taylor series and evaluate the resulting terms using Gaussian integration (Wick formula). The result will be of the form

$$V'(\varphi) = \sum_{m=0}^{\infty} \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_m\in\Lambda'} K_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)\varphi(\mathbf{x}_1)\ldots\varphi(\mathbf{x}_m)\ , \tag{7}$$

where $\Lambda' = L^{-1}\Lambda$ and $K_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ are non-local many-body interactions, each of which given by a formal perturbation series in powers of g, μ and coefficients with exponential decay in the separations $\mathbf{x}_i - \mathbf{x}_j$. The formal perturbation series diverges with the n:th Taylor coefficient growing as $n!$. The reason for this divergence can be traced back to the growth of the φ^4 interaction at infinity. Nevertheless, it turns out the series (7) *converges* provided the φ is everywhere bounded.

In the rigorous approach (5) is analysed by a cluster expansion applied to the Gaussian measure μ_Γ. This results in an expansion

$$e^{-V'(\varphi')} = \sum_{k=0}^{\infty} \sum_{\{X_1,\ldots,X_k\}} \prod_{i=1}^{k} \rho_{X_i}(\varphi')\ ,$$

where $X_i \subset \Lambda'$ are disjoint sets, called polymers and $\rho_{X_i}(\varphi')$ depends on $\varphi'|_{X_i}$. The weight ρ_X of a polymer is exponentially small in the size of X and it also carries smallness in g if $\|\varphi'|_X\|_\infty \leq R_g$ where $R_g \to \infty$ when $g \to 0$. In the case $\|\varphi'|_X\|_\infty > R_g$, ρ_X carries smallness via a factor $\exp[-cg\sum\varphi'(x)^4]$ inherited from V. These bounds guarantee that one can exponentiate the polymer expansion in any region Y where $\|\varphi'|_Y\|_\infty \leq R_g$ recovering there an expansion of the form (7) (with Λ' replaced by Y) and which is convergent.

In order to iterate this RG map one then needs to exhibit in the expansion (7) a finite number of relevant and marginal terms i.e. ones that do not contract under the linearisation of the map $V \to RV$. For the case at hand and $d = 4$ they turn out to be the local ones occuring in (6). All the other terms form an infinite dimensional space of irrelevant perturbations that contract under the linearisation. The marginal parameter g contracts due to the second order in g contributions. Finally one has to fine tune the relevant parameter μ of V so that μ will not expand and repeat this for all further iterations V_n of the RG map in order to construct the critical point, i.e. the determining of μ as a function of g so that μ_n stays small for all n. Finally, all this will work provided one can also iterate the stability bound $\exp[-cg_n\sum\varphi(x)^4]$ for the polymers in the large field region.

This procedure was carried out for the IR problem of the φ^4 problem in $d = 4$ in [33] where the Gaussian (mean field) behaviour at critical temperature

$$\langle\varphi(x)\varphi(y)\rangle \sim Z|x-y|^{-2},\ \ |x-y| \to \infty$$

was proven. A similar result was also proven in [24] using a different method.

As for the original goal of constructing a renormalisable QFT, the above analysis shows that this is not possible for the ϕ_4^4 model if we want to stay in the perturbative region during the iteration. This is so since in this case g_n is decreasing in n and for the $\epsilon \to 0$ limit we need to iterate the RG map $\log_L(\ell/\epsilon)$ times to reach the scale ℓ. Hence under this perturbative condition the $\epsilon \to 0$ limit is necessarily Gaussian. The nonperturbative result where g_n are allowed to be arbitrary was established by Aizenman and Duminil-Copin 36 years later in the beautiful work [1].

The simplest UV asymptotically free model is the two-dimensional fermionic Gross-Neveu model

$$S(\boldsymbol{\psi}) = \int (\bar{\boldsymbol{\psi}}\gamma^\mu \partial_\mu \boldsymbol{\psi} + g(\bar{\boldsymbol{\psi}}\boldsymbol{\psi})^2) d^2x, \quad \boldsymbol{\psi} = (\psi_a)_{a=1}^N.$$

Like for ϕ_4^4 the coupling λ is dimensionless, but this time in two dimensions. In [34] the free field behaviour at short distances was established

$$\langle \bar{\psi}(x)\psi(y)\rangle \sim Z(\gamma^\mu\partial_\mu)^{-1}(x-y), \;\; |x-y| \to 0.$$

The fermionic model is technically much nicer as the large field problem dos not appear (due to Pauli exclusion) and the representation (7) holds everywhere in field space. Similar result was also proven in [25] using a different method.

The only reason for considering asymptotically free problems was that then it is possible to have all the effective Hamiltonias H_ℓ close to a Gaussian. There is however another option if the RG has a non-Gaussian fixed point close to the Gaussian one. For the Gross-Neveu model this can be arranged by changing the scaling properties of the Gaussian case by replacing the covariance in Fourier space $\gamma^\mu p_\mu/p^2$ to $\gamma^\mu p_\mu/p^{2-\epsilon}$ with $\epsilon > 0$. In [35] it was proved that the RG has a non-Gaussian fixed point where $g = \mathcal{O}(\epsilon)$. This theory is non-renormalisable in the sense of requiring counterterms for an infinite number of parameters but in the Wilsonian picture there is no obstruction to its existence. The RG approach was used to clarify other puzzles in the perturbative QFT such as the renormalon poles in the Borel summed series [36] and the meaning of the perturbative ϕ_4^4 expansion that is well defined order by order even though the theory is Gaussian [37].

The polymer expansion approach to RG was subsequently used to several problems, for a review see [3]. The formalism was extended also to the study of first order phase transitions in [32], to disordered systems in [11, 12], to Kolmogorov Arnold Moser theory in [10] and even to quantum systems in [18]. However, it has remained up to this day restricted to situations that are close to Gaussian, and a more nonperturbative formalism is still missing. This situation has its parallel in the numerical approaches to RG where it has been very hard to go beyond approximative RG schemes where the RG map is truncated in order to keep the effective actions local. Adding nonlocal corrections to such schemes has not resulted in numerical improvements unlike in the bootstrap approach where adding more input to approximations has resulted in improving results for critical exponents [56]. Hence there is still work for mathematical physicists to do in this field!

2 Conformal field theory

The construction of the Gross-Neveu model fulfilled the original aim of extending constructive field theory beyond the super-renormalisable cases. A natural next step would have been to use renormalisation group to tackle more physically relevant cases, the most important being the four dimensional Yang-Mills theory. This challenge was taken by Krzysztof's colleague from Warsaw times, Tadeusz Bałaban [2].

In fundamental physics however a paradigm shift occurred in 1984 with the first superstring revolution and Krzysztof was eager to change directions. The previous year Belavin, Polyakov and Zamolodchikov had written the fundamental paper on the structure of two dimensional conformal field theory (CFT) [4], and once it was understood that CFT is a natural building block for string theory the subject exploded. While much of this study was algebraic and had little contact to action functionals and path integrals an important exception was provided by the Wess-Zumino-Witten-Novikov (WZWN) model and the related coset models. In the simplest WZNW model [60, 50] the field $g : \Sigma \to G$ is defined on a two dimensional surface Σ and it takes values in a compact simply connected Lie Group G (e.g. $G = SU(2)$). In this setup one starts with the Wess-Zumino sigma model formally defined by the path integral for an observable $F(g)$ by

$$\langle F \rangle = \int F(g) e^{-\beta S_{WZ}(g)} Dg \ ,$$

where the Wess-Zumino action is defined by

$$S_{WZ}(g) = \ -\tfrac{i}{4\pi} \int\limits_{\Sigma} \mathrm{Tr}\big((g^{-1}\partial g) \wedge (g^{-1}\bar{\partial} g).$$

Here "Tr" is the suitably normalised invariant form on the Lie algebra $\mathcal{G}$ of G. When attempting to define the path integral one encounters UV divergences and the need for renormalisation. This model is perturbatively UV asymptotically free (in the coupling constant g defined by $\beta = 1/g^2$) and believed to be massive i.e. to have a finite correlation length (here one would consider $\Sigma = \mathbb{C}$).

It was observed in [60, 50] that one can add to this action another term sharing the global $G \times G$ symmetry given by left and right actions by G. This term is of topological nature: it is defined by picking a 3-manifold B with $\partial B = \Sigma$ and extending g to B: $\tilde{g}|_\Sigma = g$ and then setting

$$S_{top}(g) \ = \ -\tfrac{i}{12\pi} \int_B \mathrm{Tr}(\tilde{g}^{-1} d\tilde{g})^{\wedge 3} \ . \tag{8}$$

For another such extension it turns out the change of action will be given by $2\pi i n$ with n an integer and therefore $e^{-kS_{top}(g)}$ is well defined for $k \in \mathbb{Z}$, and hence we may add the term $kS_{top}(g)$ to the Wess-Zumino action and preserve the $G \times G$ symmetry. Furthermore, miraculously taking $\beta = k$ the theory is conformally invariant and the resulting WZNW model is a conformal field theory. Hence one is led to study the path integral

$$\langle F \rangle = \int F(g) e^{-kS(g)} DG$$

with the action functional

$$S(g) \ = S_{WZ}(g) + S_{top}(g). \tag{9}$$

The topological nature of the model called Krzysztof's attention since it fit well with what he had done in his thesis on the canonical formalism in field theory.

To obtain the quantum Hilbert space for the model from the Euclidean path integral one takes Σ the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The classical configuration space for a theory where the field takes values in a group G would then be a space of field configurations on the circle $\mathbb{T} = \{|z| = 1\}$ i.e. the loop group $LG = \{\gamma : \mathbb{T} \to G\}$ and upon the standard (Osterwalder-Schrader) quantisation the physical Hilbert space would be a suitable L^2 space on LG i.e. the Schrödinger wave functions ψ would be functions $\psi : LG \to \mathbb{C}$. Krzysztof then observed [28] that due to the topological term in the WZNW model it is more natural to use a holomorphic polarisation to quantise the theory whereby the wave functions turn out to be holomorphic sections of a complex line bundle $\mathcal{L}$ over the complexification $LG^{\mathbb{C}}$ of LG.

In [22] a concrete description of this line bundle was given by using the action functional (9). Indeed the standard construction in a CFT with a local action functional $S(g)$ of these wave functions is to consider functionals F depending on the field g through its restriction $g|_{\mathbb{D}}$ to the unit disk $\mathbb{D}$ which would give rise to the state

$$\psi(\gamma) = \int_{g|_{\partial\mathbb{D}}=\gamma} F(g) e^{-kS_{\mathbb{D}}(g)} Dg \ , \tag{10}$$

where the integral is over fields g on $\mathbb{D}$ with fixed boundary condition and $S_{\mathbb{D}}$ the restriction of the local action functional to $\mathbb{D}$. For WZNW, however, it is not clear how to do this restriction due to the topological term. A natural idea is to extend γ from $\partial\mathbb{D}$ to $\hat{\gamma} : \mathbb{D}^c \to G$ thereby obtaining an extension $\hat{g} : \hat{\mathbb{C}} \to G$ of g and then using $S(\hat{g})$ in (10) in place of $S_{\mathbb{D}}(g)$. To see how this definition depends on the extension $\hat{\gamma}$ one uses the fundamental identity due to Polyakov and Wiegman valid on all closed surfaces:

$$S_\Sigma(gh) = S_\Sigma(g) + S_\Sigma(h) - \Gamma_\Sigma(g, h) \ , \tag{11}$$

where

$$\Gamma_\Sigma(g, h) = \frac{i}{2\pi} \int_\Sigma \mathrm{Tr}[g^{-1}\bar{\partial} g \wedge h\partial h^{-1}]. \tag{12}$$

Thus, for another extension $\hat{\gamma}' = \hat{\gamma} h$ where $h|_{\mathbb{T}} = 1$we have

$$\psi(\hat{\gamma}') = e^{-kS_{\hat{\mathbb{C}}}(h) + k\Gamma_{\mathbb{D}^c}(\hat{\gamma}, h)} \psi(\hat{\gamma}).$$

In other words, we may identify the bundle $\mathcal{L}$ as

$$\mathcal{L} = \mathbb{C} \times \mathrm{Map}(\mathbb{D}^c \to G^{\mathbb{C}}) / \mathrm{Map}_1(\mathbb{D}^c \to G^{\mathbb{C}}) \ ,$$

where Map is a suitable space of smooth maps and Map_1 smooth maps with $h|_{\mathbb{T}} = 1$ and the equivalence is

$$(\lambda, g) \sim (e^{kS_{\hat{\mathbb{C}}}(\hat{h}) - k\Gamma_{\mathbb{D}^c}(g, h)} \lambda, gh)$$

where $\hat{h}$ is extension of h by 1 to $\mathbb{D}$.

Of course the justification of this bundle structure using the path integral is non-rigorous due to the lack of mathematical definition of the path integral but one can gain insight into the WZNW model by studying this bundle. First of all it has a group structure

$$(\lambda, g) \circ (\lambda', g') = (e^{-k\Gamma(g,g')}\lambda\lambda', gg') \;, \tag{13}$$

which indeed can be checked to project to the quotient. This allows one to identify the set $\widehat{LG}^{\mathbb{C}} := \{\mathcal{L} \setminus \text{ zero section}\}$ as the central extension by $\mathbb{C} \setminus \{0\}$ of the loop group $LG^{\mathbb{C}}$. Consider now the space $\mathcal{H}$ of holomorphic sections ψ of $\mathcal{L}$. By (13) the group $\widehat{LG}^{\mathbb{C}}$ acts on $\mathcal{L}$ from left and from right. We can then define an action of $\widehat{LG}^{\mathbb{C}} \times \widehat{LG}^{\mathbb{C}}$ on $\mathcal{H}$ by

$$\ell(\hat{g}_1)r(\hat{g}_2)\psi(g) = \hat{g}_1 \circ \psi(g_1^{-1} g g_2^{*-1}) \circ \hat{g}_2^*.$$

The infinitesimal form of this action can be calculated and it gives rise to a representation of two Kac-Moody algebras of level k on $\mathcal{H}$. In [22] the reduction of this representation was studied. The cocycle $\Gamma(g, h)$ vanishes for g holomorphic or h antiholomorphic. This means that the subgroups $L^{\pm}$ of $LG^{\mathbb{C}}$ consisting of loops γ that have an (anti) analytic extension to $\mathbb{D}^c$ are embedded in $\widehat{LG}^{\mathbb{C}}$ and it is natural to look for highest weight states under their action. In [22] it was shown that precisely the so called integrable highest weight states are in $\mathcal{H}$ therefore proving that $\mathcal{H}$ contains the (finite) direct sum of the integrable highest weight representations. In [23] this result was extended to all simple Lie Groups.

In conformal field theory to each highest weight state ψ_i there corresponds a random field $\phi_i(x)$, called the primary field, and in the path integral formulation (10) this field is a function $\phi_i(x; g)$ of g so that the state ψ corresponds to the insertion of $F(g) = \phi_i(0; g)$. Then one would be interested in constructing the correlation functions of these fields on all closed surfaces Σ:

$$\langle \phi_{i_1}(x_1) \dots \phi_{i_m}(x_m) \rangle_\Sigma := \int \prod_j \phi_{i_j}(x_j; g) e^{-kS_\Sigma(g)} Dg \;. \tag{14}$$

Now formally one can perform this path integral by cutting out discs around the points $x_i \in \Sigma$, integrating over fields inside and outside the discs and finally over the fields on the boundaries[2]. Then one ends up with

$$\langle \phi_{i_1}(x_1) \dots \phi_{i_m}(x_m) \rangle_\Sigma := \langle \otimes_j \psi_{i_j} | A_{\tilde{\Sigma}} \rangle \;, \tag{15}$$

where $\tilde{\Sigma} = \Sigma \setminus \cup_i D_i$ and the amplitude

$$A_{\tilde{\Sigma}}(\gamma_1, \dots, \gamma_m) = \int_{g|_{\partial D_i} = \gamma_i} e^{-kS_{\tilde{\Sigma}}(g)} Dg \tag{16}$$

is viewed as a section of $\otimes\mathcal{L}$ and $\langle\cdot|\cdot\rangle$ in (15) is an invariant scalar product on $\widehat{LG}^m$. The above results lead one conjecture that such a scalar product exists and the Hilbert space decomposes

[2]For an example of rigorous construction of this in the context of a the Liouville theory see [47]

into a direct sum $\oplus_i \mathcal{M}_i \otimes \tilde{\mathcal{M}}_i$ where $\mathcal{M}_i, \tilde{\mathcal{M}}_i$ are the integrable highest weight modules. One could then address the conformal bootstrap approach to evaluate (16) by cutting the surface $\tilde{\Sigma}$ into a union of pairs of pants and factoring the path integral along their boundary circles and using the resolution of identity with the above direct sum decomposition eventually reducing the correlator (15) to a product of three point functions of primary fields and representation theoretic objects (conformal blocks). In the absence of rigorous theory of the path integral Krzysztof however observed that there is an alternative route to arrive at (conjectural) exact expressions of the WZNW correlation functions, to which we now turn.

The path to WZNW correlation functions goes via the so-called coset construction of Goddard, Kent and Olive [45]. These authors observed that given a compact Lie group G and a subgroup H one can obtain representations of Virasoro algebras by decomposing the representations of the group G Kac-Moody algebra into factor representations of Kac-Moody algebra based on the subgroup H. In particular representations occurring in the minimal models (e.g. the Ising model) can be obtained by taking $G = SU(2) \times SU(2)$ and H the diagonal $SU(2)$ subgroup. Hence the question arises whether there exists a full conformal field theory corresponding to such pair and what is its connection to the WZNW model.

This question was addressed in [39] where a path-integral construction of the coset CFTs was accomplished. One takes as a starting point the following action functional

$$S(g, A) = S(g) + \frac{i}{2\pi} \int_\Sigma Tr[A^{10} g^{-1} \bar{\partial} g + g \partial g^{-1} A^{01} + g A^{10} g^{-1} A^{01} - A^{10} A^{01}] \; , \tag{17}$$

where $A = A^{10} + A^{01}$ is a $\mathcal{H}$ valued gauge field with A^{10} and A^{01} the dz and $d\bar{z}$ parts respectively. It turns out the integral over A is exactly (formally) computable. In the simplest case of $\Sigma = \hat{\mathbb{C}}$ the trick is to parametrise A by $h : \hat{\mathbb{C}} \to H$ by $A^{01} = h^{-1} \bar{\partial} h$ whereby using the Polyakov-Wiegman formula (12) the action becomes

$$S(g, A) = S(hgh^*) - \ell S(hh^*).$$

Here ℓ is an integer (the Dynkin index of imbedding of $\mathcal{H}$ to $\mathcal{G}$). The Jacobean $\mathcal{J}$ of the change of variables $A \to h$ is given in terms of a determinant $\det D_A^* D_A$ where $D_A = \bar{\partial} + [A^{01}, \cdot]$ which again (by a chiral anomaly calculation) is formally explicitly given by

$$\mathcal{J} \propto \exp(2h^v k S(hh^*)) \; ,$$

where $h^v \in \mathbb{N}$ (the dual Coxeter number of H). Hence upon a final change of variables $g \to h^{-1} g h^{*-1}$ the partition function factorises

$$\int e^{-kS(g,A)} DgDA = \int e^{-kS(g)} Dg \int e^{\tilde{k} S(hh^*)} Dh$$

to a product of the group G WZNW partition function and the partition function of a QFT of a sigma model on the symmetric space $H^{\mathbb{C}}/H$; we denoted $\tilde{k} = \ell k + 2h^v$.

Now the remarkable fact is that the h integral can be explicitly calculated by Gaussian integration. Let us consider the simplest case of $H = SU(2)$. Using a Borel decomposition $h = bk$ where b is upper triangular and $k \in SU(2)$ the action becomes

$$S(hh^*) = -\frac{1}{\pi} \int (\partial_z \phi \partial_{\bar{z}} \phi + (\partial_z + \partial_z \phi) \bar{v} (\partial_{\bar{z}} + \partial_{\bar{z}} \phi) v) d^2 z \; ,$$

and the Haar measure becomes $dh = d\phi d^2 v dk$. $\phi \in \mathbb{R}$ and $v \in \mathbb{C}$ are global coordinates of the three dimensional hyperbolic space $SL(2,\mathbb{C})/SU(2)$. Thus the v integral is indeed Gaussian producing a functional determinant which quite miraculously turns out to be Gaussian in ϕ thereby reducing ϕ also to the Gaussian free field! In [39] it was shown that a similar reduction to Gaussian integrals generalises to all compact groups G by using again a Borel decomposition of $G^{\mathbb{C}} = BG$ where the Borel subgroup B is the analog of the upper triangular matrices. Performing these calculations on surfaces of genus > 0 results in to a coupling between the G and the H QFTs via nontrivial moduli of flat $\mathcal{H}$ connections $\tilde{A}$ in the parametrisation

$$A^{01} = \tilde{A}^{01} + h^{-1}\bar{\partial}h. \tag{18}$$

This way in [39] it was shown the partition function on the torus is given as a sesquilinear combination of the so called branching functions of the embedding of the two Kac-Moody algebras as predicted by [45].

The action (17) with A now a G connection serves as a starting point in the study of the WZNW correlation functions (14). Let $D_R(g)$ be the matrix of an irreducible representation of G acting in the vector space V_R. Then we are interested in the correlation functions

$$\Gamma(\mathbf{z}, \mathbf{R}, A) = \int \otimes_i D_{R_i}(g(z_i)) e^{-kS(g,A)} DG \in \otimes_i V_{R_i} \otimes V_{\bar{R}_i} \ , \tag{19}$$

where $\bar{R}$ is the complex conjugate representation. For $A = 0$ these are just the correlation functions of primary fields of the WZNW model. Using once again the Polyakov-Wiegman formula (12) one derives the global Ward identity

$$\Gamma(\mathbf{z}, \mathbf{R}, {}^h A) = e^{kS(h^{-1}h^{*-1})} \otimes_i D_{R_i}(h(z_i)) \otimes D_{\bar{R}_i}(h(z_i)) \Gamma(\mathbf{z}, \mathbf{R}, A). \tag{20}$$

This and the expectation that in a CFT the correlations factorise to sums of products of holomorphic and antiholomorphic functions motivate the definition [29] of a Kac-Moody block as a holomorphic map $A^{01} \to \gamma(A^{01}) \in \otimes_i V_{R_i}$ satisfying

$$\gamma({}^h A^{01}) = e^{S(h^{-1},\ A^{01})} \otimes_i D_{R_i}(h(z_i)) \gamma(A^{01}). \tag{21}$$

This equation defines a space $W(\Sigma, \mathbf{z}, \mathbf{R})$ of holomorphic sections of a vector bundle with fiber $\otimes_i V_{R_i}$ over the set of gauge orbits $\mathcal{A}$ i.e. the set of connections A^{01} modulo the $G^{\mathbb{C}}$ chiral gauge transformations. This space of sections is finite dimensional and it is also the space Schrödinger states of the three dimensional Chern-Simons topological QFT [40]. The latter is a theory of a gauge field a on a 3-manifold $\mathcal{M}$ with action

$$S(a) = \frac{k}{4\pi} \int_{\mathcal{M}} Tr(a \wedge da + \tfrac{2}{3} a \wedge a \wedge a).$$

Taking $\mathcal{M} = \Sigma \times \mathbb{R}$ with Σ a closed surface the quantisation of this theory [61] yields wave functions $A^{01} \in \mathcal{A} \to \psi(A^{01})$ satisfying (21). To make this a Hilbert space we need a scalar product and this is formally given by

$$\|\psi\|^2 = \int_{\mathcal{A}} |\psi(A^{01})|^2 e^{-\frac{ik}{2\pi}\int Tr(A^{01})^* \wedge A^{01}} DA.$$

Using the parametrisation (18) and the transformation law (21) this integral may be expressed [29] explicitely in terms of h and then computed in terms of iterated Gaussian integrals and eventually reduced to finite dimensional integrals involving the covariances of the Gaussian fields.

This scalar product is basic input of the Chern-Simons QFT but what is its role in the WZNW theory? This was addressed by Krzysztof in [29]. His answer to this was simple and ingenious. Let e_A be the evaluation map $\gamma \in W(\Sigma, \mathbf{z}, \mathbf{R}) \to \gamma(A^{01})$. Then

$$\Gamma(\mathbf{z}, \mathbf{R}, A) = \|e_A\|^2 e^{\frac{ik}{2\pi} \int Tr(A^{01})^* \wedge A^{01}} ,$$

where the norm is defined by duality. Hence in particular $\Gamma(\mathbf{z}, \mathbf{R}, o) = \|e_0\|^2$ and this expression is given in terms of the $G^{\mathbb{C}}/G$ model and therefore the path integral is computable by Gaussian integration. The end result is expressed in terms of finite dimensional integrals.

Working all this out for general surface Σ is a formidable tour de force carried out in [30]. There the general convergence analysis of the finite dimensional integrals was left open but strong arguments were given that they converge if and only if the fusion rules of WZNW model are satisfied. As for the questions of rigour one can not do better than cite Krzysztof: ”This is a non-rigorous work in its manipulation of formal functional integrals which lead, in the end, to a chain of Gaussian integrations. Handling these integrals required, nevertheless, careful treatment. As a result, the paper employs relatively sophisticated mathematical tools. It may be viewed as a piece of ”theoretical mathematics” in the sense of A. Jaffe and F. Quinn: it uses formal functional integral to extract an interesting mathematical structure which should be submitted now to rigorous analysis”.

The $G^{\mathbb{C}}/G$ model has also independent interest in several applications. For $G = SU(2)$, $G^{\mathbb{C}}/G = \mathbb{H}^3$, the 3-dimensional hyperbolic space, and the CFT provides an example of the AdS/CFT correspondence [55, 57]. Finally a further coset by $\mathbb{R}$ i.e. $SL(2,\mathbb{C})/(SU(2) \times \mathbb{R})$ is the simplest model of strings moving in (Euclidean) black hole geometry. Another interesting direction is the relation between the $\mathbb{H}^3$ CFT and the Liouville CFT. The conjectured correlation functions of the $\mathbb{H}^3$ CFT on $\hat{\mathbb{C}}$ can be expressed in terms Liouville theory correlation functions [58, 49]. This conjecture has interesting implications related to the analytic Langlands correspondence [59, 20] in the semiclassical limit of the Liouville CFT which corresponds to the critical level $k \to 2$ limit of the $\mathbb{H}^3$ theory.

Hence, I think it would be important if Krysztof’s call for rigorous analysis were picked up by mathematical physicists.

3 Turbulence

After his long project on WZNW models in mid 90’s Krzysztof was looking for new challenges and a change of topics. This came about at a semester on PDEs I organised at the Mittag-Leffler Institute in the fall 1994. Present in that semester were Uriel Frish and Itamar Procaccia and their debates on the Kraichnan model served as an inspiration for Krzysztof and me to venture into the field of turbulence.

In 1941 A.N. Kolmogorov argued that in fully developed turbulence a range of spatial scales exists where the velocity field exhibits approximate scale invariance properties. Furthermore this range tends to infinity as the IR and UV cutoffs provided by the scale of energy input and viscous dissipation are taken to infinity and zero respectively and in that limit a fully scale invariant stationary behaviour emerges. Later experimental and numerical studies gave evidence that there are corrections to the scaling exponents predicted by Kolmogorov and this posed a theoretical challenge to explain the observed anomalous scaling. While quantitative theoretical arguments for Navier-Stokes turbulence have been hard to come up with, in 1994 it was observed by several people [52, 53, 54] that a simple advection equation introduced in 1968 by R. Kraichnan [51] would serve as a toy model to study the violation of scale invariance.

The Kraichnan model is a special case of an advection equation describing the transport of a scalar quantity $T(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ (e.g. temperature of a fluid or an impurity concentration) by a velocity field $\mathbf{v}(t, \mathbf{x})$:

$$\partial_t T = \kappa \Delta T + \mathbf{v} \cdot \nabla T + f. \tag{22}$$

Here κ is a diffusion coefficient and $f(t, \mathbf{x})$ is a source term for T. For transport in real fluids $\mathbf{v}$ would be a solution to the Navier-Stokes equations.

In the fully developed turbulence situation the universal scale invariance properties of $\mathbf{v}$ are expected to show up in the statistics of the *velocity differences* at nearby spatial locations $\mathbf{x}, \mathbf{y}$: let $\delta_r \mathbf{v} = \mathbf{n} \cdot (\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y}))$ where $\mathbf{n} = (\mathbf{x} - \mathbf{y})/r$ and $r = |\mathbf{x} - \mathbf{y}|$. Kolmogorov's scaling theory predicts that the moments of the $\delta_r \mathbf{v}$, the velocity *structure functions*, scale as

$$S_n(\mathbf{x}) := \langle (\delta_r \mathbf{v})^n \rangle = C_n r^{n/3} (1 + \mathcal{O}(\rho/r) + \mathcal{O}(r/R)) \ , \tag{23}$$

where $\langle \cdot \rangle$ is an ergodic average. Here ρ is the scale where the viscous forces act and $\rho \to 0$ in the limit of vanishing viscosity of the fluid. The forcing scale R is the characteristic scale (e.g. size of an obstacle to the flow) where the external force acts. Thus in the so called *inertial range* of scales $\rho << r << R$ the theory predicts scale invariance of the velocity differences. Experimentally the prediction (23) appears to be violated for $n > 3$ and one rather expects to have in the limit of vanishing viscosity

$$S_n(\mathbf{x}) = D_n (\tfrac{R}{r})^{\gamma_n} r^{n/3} (1 + \mathcal{O}(r/R)) \tag{24}$$

with $\gamma_n > 0$ for $n > 3$. The theoretical explanation of this violation is one of the major open problems of turbulence.

In the Kraichnan model $\mathbf{v}$ is taken random with Gaussian statistics sharing some of the above features of a turbulent velocity field. $\mathbf{v}$ is incompressible i.e. $\nabla \cdot \mathbf{v} = 0$ with mean zero and covariance

$$\langle v_i(t, \mathbf{x}) v_j(s, \mathbf{y}) \rangle = D_{ij}(\mathbf{x} - \mathbf{y}) \delta(t - s)$$

where the spatial part has Fourier transform

$$\hat{D}_{ij}(\mathbf{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j) |\mathbf{k}|^{(\xi - d)} \chi_{\rho, R}(|\mathbf{k}|) \ , \tag{25}$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, $\chi_{\rho, R}$ is a smooth indicator of the interval $[R^{-1}, \rho^{-1}]$, and $\xi > 0$. The first factor on the RHS implies $\mathbf{v}$ is incompressible. Writing

$$D_{ij}(\mathbf{x} - \mathbf{y}) = D_{ij}(0) + d_{ij}(\mathbf{x} - \mathbf{y})$$

one has in the limit $\rho \to 0$ that

$$D_{ij}(0) = c_1 R^{\xi} \delta_{ij} \tag{26}$$

blows up as as $R \to \infty$, whereas the limit

$$\lim_{R\to\infty} \lim_{\rho\to 0} d_{ij}(\mathbf{x}) = c_2((2+\xi)\delta_{ij} - \xi \hat{x}_i \hat{x}_j)|\mathbf{x}|^{\xi} \tag{27}$$

exists, leading to

$$\langle \delta_r \mathbf{v}(t) \delta_r \mathbf{v}(s) \rangle = c_3 r^{\xi} \delta(t-s)$$

in that limit. Thus the Kraichnan velocity field mimics turbulent velocities with the scaling exponent ξ as a parameter. The property of decorrelation in time is of course a drastic simplification and a reason why the model is tractable.

Finally, $f(t, \mathbf{x})$ in (22) is a Gaussian source in large spatial scales $\sim L$ with

$$\langle f(t,\mathbf{x}) f(s,\mathbf{y}) \rangle = C_L(\mathbf{x}-\mathbf{y})\ \delta(t-s)$$

and $C_L(\mathbf{x}) = \chi(\frac{|\mathbf{x}|}{L})$ with χ a smooth bump around 0.

The equation (22) is a linear stochastic PDE for T with dissipation and it has a stationary state whose correlation functions we would like to study. In particular, in parallel to the Kolmogorov theory of turbulence we are interested in universal properties of this state in the limit $\rho, \kappa \to 0$ and $R, L \to \infty$. Universality here means independence of the details of the forcing covariance C. Also in parallel to the Kolmogorov theory the *structure functions*

$$S_n(r) := \langle (T(\mathbf{x}) - T(\mathbf{y}))^n \rangle \tag{28}$$

with $r = |\mathbf{x}-\mathbf{y}|$ are expected to exhibit universal scaling properties. Indeed if one argues along the lines of the Kolmogorov theory one predicts simple scaling $S_n(r) = c_n r^{n(2-\xi)}$.

In our work [38] it was shown this simple scaling does not hold and the structure functions exhibit anomalous scaling as in (24) once all the cutoffs except L are removed:

$$S_n(r) = E_n(\tfrac{L}{r})^{\delta_n} r^{n(2-\xi)}(1 + \mathcal{O}(r/L)) \tag{29}$$

with a nontrivial set of exponents $\delta_n > 0$ for $n > 2$ that are universal whereas the constants E_n are not. Thus the anomalous scaling is present in the statistics of the scalar even though the the advecting velocity field does not have this property. The result in [38] is based on a perturbative argument in the parameter ξ close to zero. In [15] another perturbative calculation of δ_n was given where the small parameter is $1/d$ where d is the spatial dimension.

The argument in [38] is based on exact partial differential equations satisfied by the the correlation functions of T obtained by applying Ito formula. For the one-point function the equation reads

$$\partial_t \langle T(t,\mathbf{x}) \rangle = (\kappa + \tfrac{1}{6}\,\mathrm{Tr}\, D(0)) \Delta \langle T(t,\mathbf{x}) \rangle\ , \tag{30}$$

so that the diffusion constant κ gets renormalised by the *eddy diffusivity* $\frac{1}{6}\operatorname{Tr} D(0)$, which by (26) diverges as the IR cutoff of the velocity field $R \to \infty$. Hence the large scales of the velocity field enhance strongly the diffusivity of the scalar. For the two-point function $G_2(t, \mathbf{x}-\mathbf{y}) := \langle T(t,\mathbf{x})T(t,\mathbf{y})\rangle$ this eddy diffusivity drops out (due to translation invariance), and we get

$$\partial_t G_2(t,\mathbf{x}) = M_2 G_2(t,\mathbf{x}) + C_L(\mathbf{x}) \tag{31}$$

with the elliptic differential operator

$$M_2 = 2\kappa\Delta + d_{ij}(\mathbf{x})\partial_i\partial_j\ , \tag{32}$$

so that the stationary solution is $G_{2,L}(\mathbf{x})$ with

$$G_{2,L} = M_2^{-1} C_L \tag{33}$$

The spatial homogeneity of the $d_{ij}(\mathbf{x})$ in (27) implies this solution has a scaling regime $r_\kappa << r << L$ where $r_\kappa \sim \kappa^{1/\xi}$ tends to zero as $\kappa \to 0$. One obtains in that limit

$$G_{2,L}(\mathbf{x}) = aL^{2-\xi} + \tfrac{1}{2} b|\mathbf{x}|^{2-\xi}(1 + \mathcal{O}(|\mathbf{x}|/L))$$

where the constant b is universal, proportional to the energy dissipation rate (see below). Hence, the two-point function is sensitive to the large scales of the advecting velocity field but the two-point structure function $S_2(\mathbf{x}) = \langle (T(\mathbf{x}) - T(0))^2\rangle$ has a universal limit as $L \to \infty$:

$$S_2(\mathbf{x}) = b|\mathbf{x}|^{2-\xi}.$$

The next nonzero correlation function at stationarity is the four-point function

$$\langle T(t,\mathbf{x}_1), \ldots T(t,\mathbf{x}_4)\rangle := G_4(t,\mathbf{x}).$$

Again by the Ito formula

$$\partial_t G_4(t,\mathbf{x}) = M_4 G_4(t,\mathbf{x}) + G_2(t,\mathbf{x}_1 - \mathbf{x}_2) C_L(\mathbf{x}_3 - \mathbf{x}_4) + 2 \text{ pairings},$$

which leads to the stationary state

$$G_{4,L} = M_4^{-1}(G_{2,L} \otimes C_L + 2 \text{ pairings}). \tag{34}$$

Here M_4 is the elliptic differential operator

$$M_4 = -(\sum_{i=1}^{4} \kappa\Delta_i + \sum_{i<j} d(\mathbf{x}_i - \mathbf{x}_j)\cdot\nabla_{\mathbf{x}_i}\nabla_{\mathbf{x}_j})\ .$$

Again, as $L \to \infty$ we expect the leading term in (34) to be a non-universal constant not contributing to the structure function, but now the corrections are much more tricky to find. In [38] it was shown that the leading corrections contributing to S_n are zero modes of the operator

M_n with $\kappa = 0$. This operator is singular elliptic with symbol of of spatial homogeneity $2 - \xi$ and the relevant zero modes are homogenous functions of degree $(2 - \xi)n - \delta_n$ in $\mathbf{x}_1, \dots, \mathbf{x}_n$ with nonzero δ_n for $n > 2$. In [38] and [10] the δ_n were computed perturbatively in the Hölder exponent ξ. For $\xi = 0$ the operator M_n is Laplacian whose homogenous zero modes are polynomials and $\delta_n = 0$. For $\xi > 0$ δ_n were obtained by degenerate perturbation theory as

$$\delta_n = \tfrac{N(N-2)}{2(d+2)}\xi + O(\xi^2).$$

It must be stressed that even this perturbative analysis is not rigorous and a calculation (not to mention a proof) of $\delta_n \neq 0$ for all $\xi \in (0, 2)$ would be very interesting. For $\xi = 2$ the Kraichnan velocity becomes smooth and surprisingly the stationary state of the scalar is non-trivial but explicitly computable and related to a quantum mechanical problem involving the Calogero-Sutherland Hamiltonian [8].

In [8] the above analysis was interpreted in terms of the motion of Lagrangian particles in $\mathbb{R}^d$ corresponding to the advection equation (22). For n particles with positions $\mathbf{x}_i(t) \in \mathbb{R}^d$ the stochastic flow is given by

$$d\mathbf{x}_i(t) = \mathbf{v}(t, \mathbf{x}_i(t))dt + \kappa\, d\mathbf{b}(_it)\ ,$$

where $\mathbf{b}_i(t)$ are independent Brownian motions. For $\rho > 0$, $\mathbf{v}$ is smooth and the transition probability kernel $P(\mathbf{x}, \mathbf{y}, t, \mathbf{v})$ (where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$) of this SDE is defined and we have

$$\langle P(\mathbf{x}, \mathbf{y}, t, \mathbf{v})\rangle = e^{-tM_n}(\mathbf{x}, \mathbf{y})\ , \tag{35}$$

where the expectation is over $\mathbf{v}$. The RHS Lagrangian has a limit as $\rho \to 0$ and as $\kappa \to 0$. Furthermore the result is continuous in $\mathbf{x}, \mathbf{y}$ (in fact differentiable [48]). On the other hand it is formally the expectation over the velocity ensemble of the transition kernel of the ordinary differential equation

$$\dot{\mathbf{x}}_i(t) = \mathbf{v}(t, \mathbf{x}_i(t)), \quad x_i(0) = \mathbf{y}_i\ .$$

For $\rho > 0$, $\mathbf{v}$ is smooth a.s. and the solution of this equation is unique given the initial condition so if for some i, j we have $\mathbf{y}_i = \mathbf{y}_j$ then $\langle P(\mathbf{x}, \mathbf{y}, t, \mathbf{v})\rangle$ would be zero if $\mathbf{x}_i \neq \mathbf{x}_j$. However, if $\rho = 0$ the RHS of (35) does not satisfy this property and we conclude that the Lagrangian trajectories once the cutoffs are removed are not unique and in fact we expect them to be stochastic for a typical realisation of the vector field $\mathbf{v}$. Note that since the velocity field is not Lipschitz in its spatial dependence non-uniqueness of solutions is to be expected. Nevertheless the lack of uniqueness here is very striking. This phenomenon has been termed as *spontaneous stochasticity* and is believed to be an important property of fully developed turbulence. The Kraichnan model is the first nontrivial case where this phenomenon has been established.

Refinement of the analysis of the zero modes of the operators M_n led to a novel picture of the Lagrangian flow in [8]. First, the zero modes $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ are obviously conserved by the average flow since $e^{-tM_n} f = f$. Hence $\mathbb{E}^{t,\mathbf{y}} f = f(\mathbf{y})$. In [8] an infinite family of homogenous *slow modes* f of the flow was uncovered where $\mathbb{E}^{t,\mathbf{y}} f = t^{\sigma/(2-\xi)} f(\mathbf{y})$ with σ the homogeneity degree of f. These functions have super-diffusive growth and they were shown to control the asymptotic probabilities of the Lagrangian particles coming together.

Returning to the advection equation (22) and defining "energy" $E(t, \mathbf{x}) = \frac{1}{2}\langle T(t, \mathbf{x})^2 \rangle$ Ito formula gives

$$\dot{E} = -\kappa \langle (\nabla T)^2 \rangle + \tfrac{1}{2} C(0) \ ,$$

so that we have input of energy $\frac{1}{2}C(0)$ into the system and its dissipation by the diffusion. In a stationary state they balance: $\kappa\langle(\nabla T)^2\rangle = \frac{1}{2}C(0)$. In particular the dissipation $\kappa\langle(\nabla T)^2\rangle$ does not vanish as $\kappa \to 0$, a phenomenon coined as dissipative anomaly. Looking at this energy balance scale by scale [38] reveals a constant flux of energy from large scales L where the energy is pumped into the system to the dissipation scale $\sim \kappa^{1/\xi}$ where it is dissipated. This is the same phenomenology as one expects in Navier-Stokes turbulence.

The discussion so far dealt with incompressible velocities. One may include compressibility in the Kraichnan model by adding to the prefactor in (25) a term $\lambda \hat{k}_i \hat{k}_j$ with $\lambda > 0$. In [43] an interesting phase transition in the parameter λ was uncovered. For small λ the explosive separation of the Lagrangian trajectories persists but at strong compressibility the trajectories implosively collapse together. Correspondingly at weak compressibility, the scalar exhibits the above direct cascade of the energy to small scales, anomalous scaling of structure functions and dissipative anomaly. In contrast at strong compressibilty one has an inverse cascade of energy to large scales. The anomalous scaling, present in scales smaller than L is absent in larger scales and the dissipative anomaly is absent.

4 Later work

In these notes I have covered some of the work Krzysztof did before 2000. The bulk of it was done while he was at IHES during 1981-2001. In 2001 he moved to ENS Lyon where he started to have more PhD students and also to teach, both of which he had been missing at IHES. During this period he continued working on WZNW models and turbulence but remarkably he also embarked on new projects in quite different fields. In WZNW models he extended the previous work to boundary CFT setup i.e. to a formulation of the theory on a surface Σ with a boundary and equipped with conformal boundary conditions. The intricacies of the topological term in the action had been addressed by him already much earlier in [28] where a cohomological description was given. A geometrical reformulation of this in the language of bundle gerbes was given in [42]. Surprisingly, he found use of this formalism later in the construction of topological invariants for condensed matter systems [13, 14]. From his work on turbulence Krzysztof was led to other questions in non-equlibrium systems. In [16] an elegant and comprehensive approach to various fluctuation relations for classical nonequilibrium dynamics described by diffusion processes is presented. These relations compare the statistics of fluctuations of the entropy production or work in the original process to the similar statistics in the time-reversed process. The origin of a variety of fluctuation relations is traced to the use of different time reversals. Again Krzysztof approached these questions with his characteristic thoroughness and this led even to an experimental work where he was heavily involved [46]. There a modified fluctuation-dissipation-theorem was experimentally verified by studying the position fluctuations of a colloidal particle whose motion is confined in a toroidal optical trap. The theoretical interpretation of this setup was an equilibrium-like fluctuation-dissipation relation in the Lagrangian frame of the mean local velocity of the particle. Finally Krzysztof embarked on yet another new project where he was able to combine his mastery of conformal field theory and non-equilibrium physics to the study of quantum non-equilibrium systems: heat transport in the Luttinger model [41] and full-counting statistics of energy transfers in unitary Conformal Field Theories [31]. This project was interrupted by his untimely death in January 2022. His passing away is a big loss to the mathematical physics community and a personal loss to many of us, his friends and colleaugues.

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Nikita Nekrasov Awarded 2023 Dannie Heineman Prize for Mathematical Physics

WASHINGTON, Nov. 10, 2022 – AIP and the American Physical Society are pleased to announce Nikita Nekrasov as the recipient of the 2023 Dannie Heineman Prize for Mathematical Physics "for the elegant application of powerful mathematical techniques to extract exact results for quantum field theories, as well as shedding light on integrable systems and noncommutative geometry."

The annual award acknowledges significant contributions to the field of mathematical physics and will be presented at an upcoming APS meeting.

"We are so pleased to recognize Nikita Nekrasov with this award," said Michael Moloney, CEO of AIP. "His work has taken abstract principles in mathematics and proved them essential for theoretical physics, building upon our fundamental knowledge of how the universe works – the pondering on which has been an inspiration to generations of scientists."

Nekrasov, a professor at Stony Brook University's Simons Center for Geometry and Physics and Yang Institute for Theoretical Physics, used techniques from topology to solve important problems in theoretical physics, namely, exactly calculating the effects of the strong force holding together nuclei.

Complex problems in quantum physics are often broken into two pieces: an explicit solution of a simpler system, and the analysis of a "perturbation" that reflects the small difference of a realistic model from that simple system. As an example, in a simplified picture, freely propagating particles occasionally meet and interact with other particles along their way. Having many successive interactions is less likely, which makes the perturbation terms mathematically manageable. However, some natural phenomena, such as the strong force, do not follow this rule and require a different approach.

"One needs better understanding of how to account for the effects of strong force," said Nekrasov. "I found a class of theories for which this can be done exactly, but you have to bring in a novel type of mathematics: topology and non-commutative geometry."

The mathematics can also be used for exactly solvable models describing many-body interactions, be it planets in the solar system, cold atoms, or electrons in a quantum Hall effect. Nekrasov discovered that, under the assumption of supersymmetry, the mathematics of strong interactions is the same as the mathematics describing many particles living on a line and interacting with some repulsive force.

"Instead of trying to visualize the quarks and gluons inside an atomic nucleus, which we cannot see directly, you could set up a laboratory with quantum wires, do some measurements, and then try to translate that result to the world of elementary particles," Nekrasov said. "That's the amazing fact about physics and mathematics. There are unexpected connections between different fields."

A French-Russian national, Nekrasov grew up in Russia, where he became hooked on string theory and mathematical physics after reading a *Scientific American* article by Professor Michael Green (recipient of the 2002 Dannie Heineman Prize for Mathematical Physics). He earned his doctorate at Princeton University and completed a postdoctoral fellowship at Harvard University as a Junior Fellow at the Harvard Society of Fellows. After briefly returning to Princeton University as a Dicke Fellow, he became professor at the Institut des Hautes Études Scientifiques in France. Since 2013 he has been a professor at the Simons Center for Geometry and Physics and Yang Institute for Theoretical Physics at Stony Brook University.

"It's an honor to receive this award, and in some sense, it's a way to shake hands with a lot of my heroes, the people who inspired me in my work," said Nekrasov.

Nekrasov hopes to continue connecting abstract mathematics to theoretical physics and is currently interested in finding applications of quantum field theory to number theory.

ABOUT THE HEINEMAN PRIZE

The Heineman Prize is named after Dannie N. Heineman, an engineer, business executive, and philanthropic sponsor of the sciences. The prize was established in 1959 by the Heineman Foundation for Research, Education, Charitable and Scientific Purposes, Inc.The prize will be presented by AIP and APS on behalf of the Heineman Foundation at a future APS meeting. A special ceremonial session will be held during the meeting, when Nekrasov will receive the $10,000 prize. http://www.aps.org/programs/honors/prizes/heineman.cfm

ABOUT AMERICAN INSTITUTE OF PHYSICS

The American Institute of Physics (AIP) is a 501(c)(3) membership corporation of scientific societies. AIP pursues its mission – to advance, promote, and serve the physical sciences for the benefit of humanity – with a unifying voice of strength from diversity. In its role as a federation, AIP advances the success of its Member Societies by providing the means to pool, coordinate, and leverage their diverse expertise and contributions in pursuit of a shared goal of advancing the physical sciences in the research enterprise, in the economy, in education, and in society. In its role as an institute, AIP operates as a center of excellence using policy analysis, social science, and historical research to promote future progress in the physical sciences.

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Award Deadlines

Nominations for the 2024 Heineman Prize are due by 1 June, 2023.
For more information and instructions go to

https://www.aps.org/programs/honors/prizes/heineman.cfm

Nominations for the IOP Quantum Awards are due by 15 March, 2023.
For more information and instructions go to

https://ioppublishing.org/two-international-quantum-awards-now-open-for-nominations

Time's Arrow

Scientific anniversaries

1873. Josiah Willard Gibbs published two foundational works in statistical mechanics.

1873. James Clerk Maxwell published his "Treatise on Electricity and Magnetism."

1923. Arthur Eddington published "The Mathematical Theory of Relativity."

Lost luminaries

Howard Weiss, 5 November 2022.

Huzihiro Araki, 16 December, 2022. Araki, the second President of IAMP, will be commemorated in a future issue.

Readers are encouraged to send items for "Time's Arrow" to bulletin@iamp.org.

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. PROFESSOR ANDANG SUNARTO, Uversitas Islam Negeri Fatmawati Sukarno Bengkulu, Indonesia
2. MS. HASNA JABOUR, Université de Paris, France
3. PROFESSOR FEDERICO BONETTO, Georgia Institute of Technology
4. PROFESSOR MUHAMMAD KAMRAN, COMSATS University Islamabad, Wah Campus, Pakistan
5. DOCTOR JUNCHEN RONG, IHES, Bures-sur-Yvette, France
6. DOCTOR BENOIT SIROIS, Ecole normale superieure, Paris, France

Recent conference announcements

Quantum Transport: Disorder, Interactions and Integrability

January 26-27, 2023, Accademia Nazionale dei Lincei, Roma, Italy.

Universality in Condensed Matter and Statistical Mechanics

February 6-8, 2023, Univ. Roma Tre, Roma, Italy.

Research term on Quantum Information Theory

February 27- March 3, 2023, at Ignacio Cirac Lab - ICMAT Madrid, Spain.

School: Scaling limits and generalized hydrodynamics

March 27- 31, 2023, at Gran Sasso Science Institute, l'Aquila, Italy.

Correlations in Mathematical Quantum Mechanics

June 21-23, 2023, at Copenhagen University.

Seminar series

Mathematical Challenges in Quantum Mechanics - Online Seminars

For an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

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