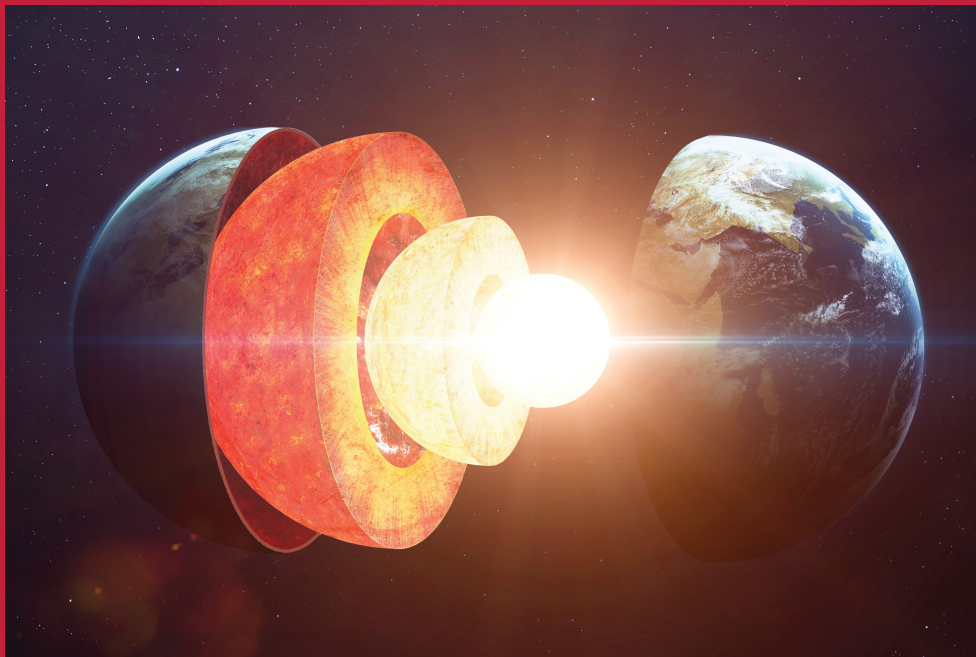


IAMP News Bulletin

July 2021



International Association of Mathematical Physics

International Association of Mathematical Physics News Bulletin, July 2021

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Cover picture: The inverse problem for the inner structure of the earth.



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News Bulletin (International Association of Mathematical Physics)

Welcome to ICMP

On behalf of the International Association of Mathematical Physics and of the Local Organizing Committee, we would like to welcome participants to the XX International Congress on Mathematical Physics!

The program of the conference and more information about ICMP (and about the Young Researcher Symposium preceding ICMP) are available at

<https://www.icmp2021.com>

As usual, ICMP will start on Monday, August 2, with the Opening Ceremony, where recipients of the Henri Poincaré Prize (sponsored by the Daniel Iagolnitzer Foundation), of the Early Career Award (sponsored by Springer) and of the IUPAP Young Scientist Prize in Mathematical Physics will be announced. The scientific program of the Congress consists of 16 plenary talks, 70 invited and 81 contributed talks divided in 12 thematic sessions. Additionally, on Monday, August 2, at 5:30pm, we will have a public lecture by Michel Mayor (2019 Nobel Prize Laureate for the discovery of an exoplanet). On Friday, August 6, over lunch, we will also have a human right session. This year, the human-rights session is devoted to the issue of gender equality and the underrepresentation of women in science. Finally, as part of the social program, available for on-site but also for online participants, we will offer the possibility of virtual guided visits of CERN.

This year, because of the covid-19 pandemic, YRS and ICMP will take place in hybrid form, with some participants in Geneva and others connected online (on-site and online registration are still possible at the conference webpage mentioned above). While we are sorry that we cannot welcome everybody here in Geneva, the hybrid format gives us the possibility to reach more members of our community.

If you plan to travel to Geneva, you can check covid-19 related restrictions for entering Switzerland at

<https://travelcheck.admin.ch/home>

In general, vaccinated visitors can travel to Switzerland with no further restriction (but they may need to fill an entry form and, of course, they are subject to all general entry requirements). Also unvaccinated participants can enter Switzerland (possibly providing a negative PCR test), if they come from most European countries (not the UK) or from some non-European countries, including the US, Israel, Japan, Australia and New Zealand (this list may change).

We are looking forward to welcoming you in Geneva or to meeting you online at YRS and ICMP!

ANTON ALEKSEEV and BENJAMIN SCHLEIN

Inverse Problems

by GUNTHER UHLMANN (Seattle)



As reported in the *January issue*, the 2021 AMS-SIAM George David Birkhoff Prize in Applied Mathematics was awarded to Gunther Uhlmann “for his fundamental and insightful contributions to inverse problems and partial differential equations, as well as for his incisive work on boundary rigidity, microlocal analysis and cloaking. Uhlmann’s work is distinguished by its mathematical beauty and relevance to many significant applications, especially in medical imaging, seismic prospecting and general inverse problems.” The News Bulletin is pleased to publish a review article by Uhlmann related to the work for which he won the Birkhoff Prize.

1 Introduction

In inverse problems causes are sought for an observed or desired effect. They arise in all areas of science and technology. The mathematical study of inverse problems has a long history. Newton, as shown in his masterpiece *Principia*, found the inverse square of the distance law for the gravitational force in explaining Kepler’s laws. Of course he invented calculus in the process! In this case some of the solutions of an ordinary differential equation are known and we are looking for the form of the underlying force producing the observed solutions of the ordinary differential equation. Sonar and radar are by now standard applications of inverse scattering: we send acoustic waves in the case of sonar or electromagnetic waves in the case of radar and measure the response to determine the location and shape of a target. In this article I have chosen to mention a few inverse problems on which I have worked. These include X-ray tomography, electrical impedance tomography, also called Calderón’s problem, boundary rigidity and travel-time tomography, some inverse problems arising in general relativity, and inverse problems in nonlinear acoustics.

Medical imaging is one of the important areas of applications of inverse problems. The motivation is to develop non-invasive methods to make an image of the interior of the body using electromagnetic waves, acoustic waves, or elastic waves. An important example that revolutionized the practice of medicine is the development of CT scans, that we describe in more detail.

2 X-ray tomography (CT-scans)

Central figures in this subject are J. Radon, A. Cormack and G. Hounsfield. Cormack and Hounsfield were awarded the 1979 Nobel Prize in Medicine for the development of CT scans.



J. Radon

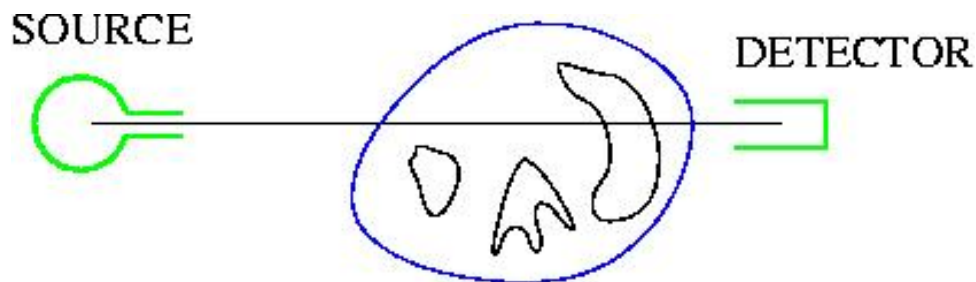


A. Cormack



G. Hounsfield

An X-ray is a high frequency electromagnetic wave that travels approximately in a straight line. We measure the intensity of the X-ray at the detector, knowing the intensity at the source. We are considering a two dimensional section of the body.



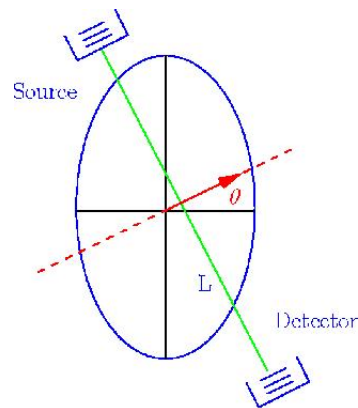
The inverse problem is whether we can recover the density of the different tissues from the attenuation of the intensity of X-rays.

A. Cormack in his Nobel lecture ([8]) stated: "If a fine beam of gamma-rays of intensity I , is incident on the body and the emerging intensity is I_0 , then the measurable quantity $g = \ln(I_0/I) = \int_L f ds$, where f is the variable absorption coefficient along the line L . Hence if f is a function in two dimensions, and g is known for all lines intersecting the body, the question is: Can f be determined if g is known? Again this seemed like a problem which would have been solved before, probably in the 19th Century, but again a literature search and enquiries of mathematicians provided no information about it. Fourteen years would elapse before I learned that Radon had solved this problem in 1917." He then explained how he derived his own reconstruction formula and tested it experimentally.

We state now Radon's inversion formula. As described by Cormack in his Nobel lecture we have that the intensity I_0 at the source and I at the receiver is given by

$$I = e^{-\int_L f} I_0.$$

This is called Beer's law in tomography. Here L is the line connecting the source and the receiver. We parametrized lines in two dimensions by the normal to the line θ and s where $|s|$ is the distance of the line to the origin.



Then the information given from the intensity of the X-ray measured at the source and receiver gives the line integral of the function f .

$$Rf(s, \theta) = g(s, \theta) = \int_{\langle x, \theta \rangle = s} f(x) dH = \int_L f.$$

Here R is the Radon transform. Radon's inversion formula is

$$f(x) = \frac{1}{4\pi^2} p.v. \int_{S^1} d\theta \int \frac{\frac{d}{ds} g(s, \theta) ds}{\langle x, \theta \rangle - s}.$$



Reconstruction of the Shepp-Logan phantom. Left: original. Right: reconstruction with the filtered backprojection algorithm in [32].

3 Electrical Impedance Tomography and Calderón's Problem



This inverse problem is to recover the electrical conductivity of a medium by making voltage and current measurements on the boundary. This is a very different method than X-ray tomography since the currents go everywhere.

Alberto P. Calderón [7] gave the first general mathematical formulation of this problem. He worked in the late 40's after receiving his engineering degree for Yacimientos Petrolíferos Fiscales, the oil company of Argentina. His motivation was oil prospection. In his Speech at Universidad Autónoma de Madrid accepting the 'Doctor Honoris Causa he stated (translation from Spanish):

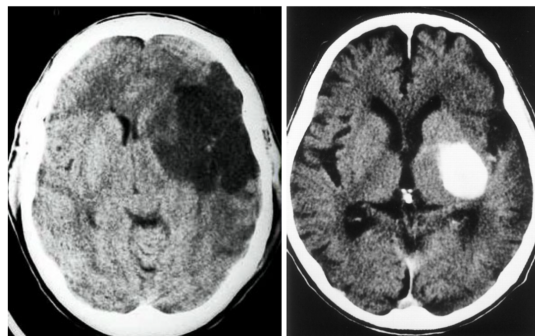
My work at "Yacimientos Petrolíferos Fiscales was very interesting, but I was not well treated, otherwise I would have stayed there."

Of course it was very fortunate for mathematics that Calderón was not treated well at YPF!

This imaging method is also called *electrical impedance tomography* (EIT) or *resistivity imaging*. One recent application has been the identification of stroke. We would like to have a portable method that can be carried in an emergency ambulance for instance, that can tell quickly whether a patient with a stroke has had an hemorrhage or a clot. The conductivity is quite different in both cases since blood has a very high conductivity.

Ischemic stroke:
low conductivity.

CT image from
Jansen 2008

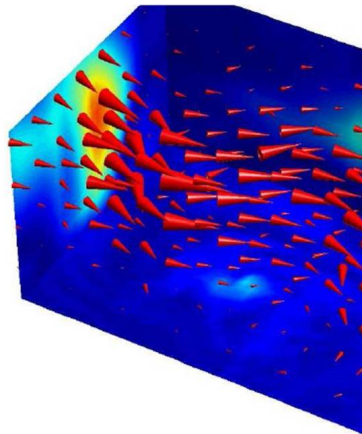


Hemorrhagic stroke:
high conductivity.

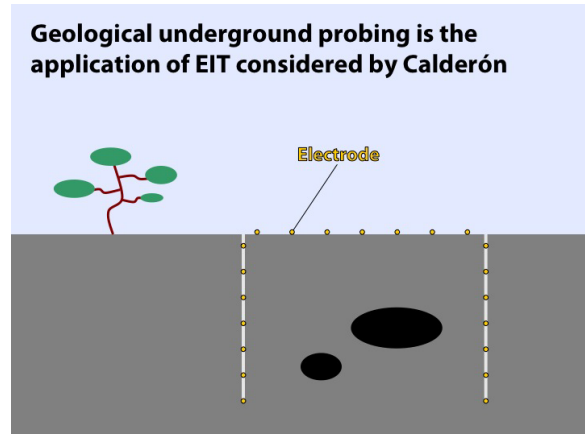
CT image from
Nakano *et al.* 2001

Same symptoms in both cases!

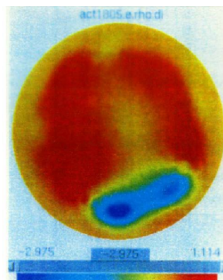
Other applications include medical imaging for instance in the monitoring of pulmonary edema [20], in early breast cancer detection [40]. In this article, we consider the problem of identification of a stroke using EIT.



Currents.



Thanks to S. Siltanen.



ACT3 imaging blood as it leaves the heart (blue) and fills the lungs (red) during systole. Thanks to D. Issacson.



Thanks to S. Siltanen.

Now we describe the mathematical problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let γ be a $C^2(\overline{\Omega})$ strictly positive function in $\overline{\Omega}$. Given a voltage potential f on the boundary the equation for the potential, under the assumption of no sources or sinks of currents, is given by

$$\operatorname{div}(\gamma \nabla u) = 0; \quad u|_{\partial\Omega} = f.$$

The Dirichlet-to-Neumann map (DN map) is defined by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}.$$

The inverse problem is to study the map

$$\gamma \xrightarrow{\Lambda} \Lambda_\gamma.$$

This is a nonlinear map. We want to study the question of injectivity of Λ^{-1} (if it exists), continuity estimates for Λ and Λ^{-1} , reconstruction of γ given Λ_γ , characterization of the range of Λ , and the development of numerical algorithms to recover γ from Λ_γ . For a survey on progress on these questions, see [37].

The question is what voltages we should put on the boundary to recover the conductivity. This will be the boundary values of complex geometric optics (CGO) solutions. The first step in the construction of CGO solutions is a reduction to the Schrödinger equation by making the substitution $w = \sqrt{\gamma}u$, where u solves $\operatorname{div}(\gamma\nabla u) = 0$ on Ω . Then w solves

$$(\Delta - q)w = 0, \quad \text{on } \Omega$$

if $w = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$.

The fundamental lemma is the existence of CGO solutions for the Schrödinger equation, which was proven in [35] and [36].

Lemma. *Let $q \in L^\infty(\Omega)$. Let $\varrho \in \mathbb{C}^n \setminus \{0\}$, $\varrho \cdot \varrho = 0$. Then $\exists M > 0$ such that if $|\varrho| \geq M$, there exists solution to $(\Delta - q)u = 0$ on Ω of the form*

$$u = e^{x \cdot \varrho} (1 + \psi(x, \varrho)),$$

with $\|\psi(\cdot, \varrho)\|_{L^2(\Omega)} \xrightarrow{|\varrho| \rightarrow 0} 0$.

Using CGO solutions the following two basic uniqueness results were proved.

Theorem. *For $n \geq 2$, let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain with smooth boundary. Let $\gamma^{(i)}$, $i = 1, 2$ be two conductivities in $C^2(\bar{\Omega})$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.*

The case of dimension three or higher is due to Sylvester and Uhlmann [35] and the two dimensional case is due to Nachman [30]. The regularity in this uniqueness result has been extended to $\gamma^{(i)} \in W^{1,n}(\bar{\Omega})$ by Haberman [17] for $n = 3, 4$. The conjecture is that $W^{1,n}(\bar{\Omega})$ is the optimum regularity for dimension three or higher. In two dimensions, the theorem was extended to conductivities $\gamma^{(i)} \in L^\infty(\bar{\Omega})$ by Astala and Päivärinta in [3].

The proof of this theorem proceeds by proving a more general result. Namely one reduces the problem to consider the set of Cauchy data for solutions of the Schrödinger equation (see [35] for more details).

Let $n \geq 2$. Let $q \in L^\infty(\Omega)$. We define the set of Cauchy data for the associated Schrödinger equation by

$$\mathcal{C}_q = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} \right) \mid (\Delta - q)u = 0 \quad \text{on } \Omega, u \in H^1(\Omega) \right\}.$$

Theorem. *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}.$$

Then $q_1 = q_2$.

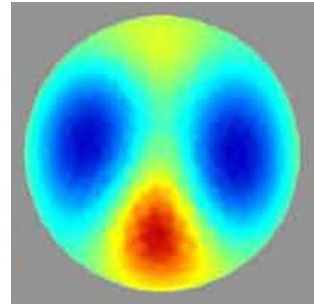
This theorem was proved in dimension three or higher by Sylvester and Uhlmann, see [35] and the two dimensional case is due to Bukhgeim, see [5].

Also, a numerical algorithm was developed in [19] in two dimensions.

We show in this article how to use these solutions for the linearized problem which is of interest in its own right.



Original



Reconstruction

Linearized Problem.

Assume $f \in L^\infty(\Omega)$ and

$$\int_{\Omega} fuv = 0,$$

for $u, v \in L^2(\Omega)$ as solutions of

$$(\Delta - q)u = (\Delta - q)v = 0, \text{ on } \Omega.$$

Does this imply that $f = 0$?

In dimension $n \geq 3$, we take

$$u = e^{x \cdot \varrho_1} (1 + \psi_1(x, \varrho_1)), \quad v = e^{x \cdot \varrho_2} (1 + \psi_2(x, \varrho_2)),$$

with $\varrho_j \in \mathbb{C}^n$, $\varrho_j \cdot \varrho_j = 0$, for $j = 1, 2$, such that

$$\varrho_1 = \eta - i\left(\frac{\xi + l}{2}\right), \quad \varrho_2 = \eta - i\left(\frac{\xi - l}{2}\right),$$

where $\eta, \xi, l \in \mathbb{R}^n$. And $\langle \eta, \xi \rangle = \langle \eta, l \rangle = \langle \xi, l \rangle = 0$ with $|\eta|^2 = \frac{|\xi|^2 + |l|^2}{4}$.

Then by letting $|l| \rightarrow \infty$ we have

$$\int_{\Omega} e^{-ix \cdot \xi} f(x) dx = 0, \quad \forall \xi \in \mathbb{R}^n,$$

which implies that $f = 0$ a.e. on Ω .

Recovering Discontinuities of Conductivities

In [14], in two dimensions, a new method was developed to determine discontinuities of a conductivity by measuring the DN map. This method has the advantage that one does not need to know a-priori the background conductivity. The other is that one can reconstruct inclusions within inclusions. As is mentioned earlier, the particular motivation of [14] is to provide for a cheap and quick method to detect at an emergency ambulance whether a patient is suffering from bleeding or a clot in the brain. The main problem is that many imaging methods have



Several jump surfaces in the presence of smooth background conductivity

difficulties seeing beyond the skull. It also can be applied to more general inclusions, for example the picture below.

We give a brief sketch of the method used in the reference [14]. For complex frequencies $\zeta = \zeta_R + i\zeta_I \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$, one can decompose $\zeta = \tau\eta$, with $\tau \in \mathbb{R}$ and $\eta = \eta_R + i\eta_I$, $|\eta_R| = |\eta_I| = 1$, $\eta_R \cdot \eta_I = 0$. Now consider solutions of the form

$$u(x) := e^{i\zeta \cdot x} w(x, \tau) = e^{i\tau\eta \cdot x} w(x, \tau).$$

Physically speaking, τ can be considered as a spatial frequency, with the voltage on the boundary $\partial\Omega$ oscillating at length scale τ^{-1} . We have

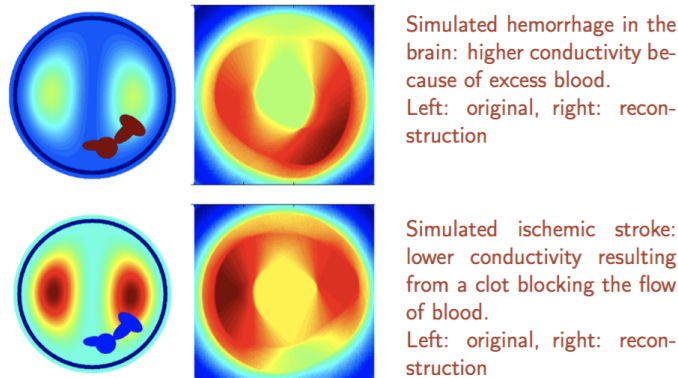
$$\Delta w(x, \tau) + 2i\tau\eta \cdot \nabla w(x, \tau) + \left(\frac{1}{\sigma} \nabla \sigma\right) \cdot (\nabla + i\tau\eta) w(x, \tau) = 0.$$

Taking the partial Fourier transform \widehat{w} in the τ variable and denoting the resulting dual variable by t , which can be thought of as a “pseudo-time,” one obtains

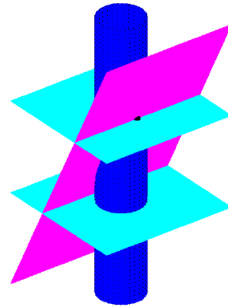
$$\Delta \widehat{w}(x, t) - 2\eta \frac{\partial}{\partial t} \cdot \nabla \widehat{w}(x, t) + \left(\frac{1}{\sigma} \nabla \sigma\right) \cdot (\nabla - \eta \frac{\partial}{\partial t}) \widehat{w}(x, t) = 0.$$

One has created an artificial time for the elliptic equation and one can study the propagation of singularities. These are two dimensional planes generated by the real and imaginary part of the complex frequency ρ that reflect at the jump of the conductivity. The reflection is measured at the boundary. See the picture below. Ongoing work is to do similar analysis in dimension three and higher.

Example simulations done in [14] are indicated below.



[14]



Several jump surfaces in the presence of smooth background conductivity

Anisotropic Conductors

In this case the conductivity depends also on direction. Muscle tissue is a prime example of an anisotropic conductor. For instance for cardiac muscle tissue the transverse conductivity is 2.3 mho while the longitudinal one is 6.3 mho. An anisotropic conductivity is modeled as a positive-definite symmetric metric $\gamma = \gamma^{ij}$. Under the assumption of no sources or sinks of currents the potential u , given a voltage f on the boundary, is given by the solution of the Dirichlet problem

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\gamma^{ij} \frac{\partial u}{\partial x_j} \right) = 0 \text{ on } \Omega; \quad u|_{\partial\Omega} = f.$$

If $\nu = (\nu^i)$ is the unit outer normal vector at $\partial\Omega$, the DN map is defined by

$$\Lambda_\gamma = \sum_{i,j=1}^n \gamma^{ij} \frac{\partial u}{\partial x_j} \nu^i.$$

If $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$ is a diffeomorphism with $\psi|_{\partial\Omega} = \text{Identity}$, then $\Lambda_{\tilde{\gamma}} = \Lambda_\gamma$, where

$$\tilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|} \right) \circ \psi^{-1} =: \psi^* \gamma.$$

Conjecture: Let $\gamma, \tilde{\gamma}$ be conductivities satisfying $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$. Then there exists a diffeomorphism $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$, with $\psi|_{\partial\Omega} = \text{Identity}$ so that $\tilde{\gamma} = \psi^* \gamma$.

The two-dimensional case of the conjecture is known for L^∞ conductivities, see [4]. In this case one can transform the anisotropic problem to an isotropic one by using isothermal coordinates and then one applies the Astala-Päiväranta result for isotropic conductivities [3]. In three dimensions or higher it is known for real-analytic conductivities [25].

It was pointed out in [27] that this can be reformulated as a geometric inverse problems in dimension three or higher. Let (M, g) be a compact Riemannian manifold with boundary. Let Δ_g be the Laplace-Beltrami operator, where in local coordinates we have

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right),$$

with g as a positive definite symmetric metric matrix and $(g^{ij}) = (g_{ij})^{-1}$ as the inverse matrix. The connection between the metric and the conductivity is given by $\gamma^{ij} = \sqrt{\det g} g^{ij}$. We solve the Dirichlet problem

$$\begin{aligned}\Delta_g u &= 0 \text{ on } M \\ u|_{\partial M} &= f.\end{aligned}$$

The DN map is given in the local coordinates by $\Lambda_g(f) = \sum_{i,j=1}^n \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g} \Big|_{\partial M}$, where $\nu = (\nu^1, \dots, \nu^n)$ is the unit-outer normal to ∂M .

Theorem ([27]). *Assume $n \geq 3$. Let $(M, g_i), i = 1, 2$, be real-analytic, connected, compact Riemannian manifolds with boundary. Let $\Gamma \subseteq \partial M$ open and nonempty. Assume*

$$\Lambda_{g_1}(f)|_{\Gamma} = \Lambda_{g_2}(f)|_{\Gamma}, \quad \forall f, f \text{ supported in } \Gamma.$$

Then there exists a real analytic diffeomorphism $\psi : M \rightarrow M$ such that

$$g_1 = \psi^* g_2, \quad \psi|_{\Gamma} = \text{Identity}.$$

In fact, one can determine the topology of M as well.

In two dimensions, there is additional obstruction because the Laplace-Baltrami operator is conformally invariant. We have

Theorem ([27]). *Let $(M, g_i), i = 1, 2$ be oriented, connected, compact Riemannian surface with boundary. Let $\Gamma \subseteq \partial M$ open and nonempty. Assume*

$$\Lambda_{g_1}(f)|_{\Gamma} = \Lambda_{g_2}(f)|_{\Gamma}, \quad \forall f, f \text{ supported in } \Gamma.$$

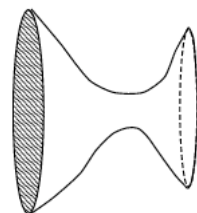
Then there exists a C^∞ diffeomorphism $\psi : M \rightarrow M$ and $\beta > 0, \beta|_{\Gamma} = 1$ such that

$$g_1 = \beta \psi^* g_2, \quad \psi|_{\Gamma} = \text{Identity}.$$

In fact, one can determine the topology of M as well.

Non-uniqueness for EIT (Cloaking)

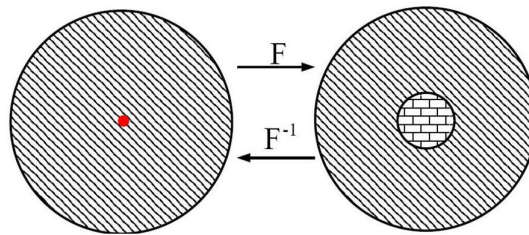
In [15], anisotropic conductivities that cannot be detected using voltage and current measurements at the boundary were constructed. They are "invisible" to electrical measurements. The method used has been called transformation optics and has been applied to cloaking for electrical magnetic waves, acoustic waves, elastic waves, etc.; see [16] for a survey.



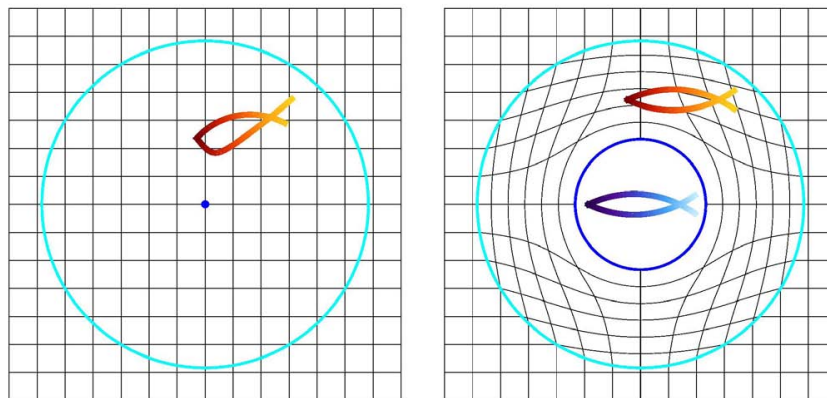
The motivation is when the bridge connecting the two parts of the manifold gets narrower, the boundary measurements give less information about the isolated area. This idea was realized in [16] in the Euclidean space as is explained below.

Let $\Omega = B(0, 2) \subset \mathbb{R}^3, D = B(0, 1)$, where $B(0, r) = \{x \in \mathbb{R}^3; |x| < r\}$. Suppose $F : \Omega \setminus \{0\} \rightarrow \Omega \setminus \overline{D}$ is a diffeomorphism with $F|_{\partial\Omega} = \text{Identity}$ given by

$$F(x) = \left(\frac{|x|}{2} + 1 \right) \frac{x}{|x|}.$$



Transformation optics.



Virtual space on the left. Physical space on the right. Picture from [26].

Let $\gamma = g = \text{identity}$ on $B(0, 2)$, $\hat{\gamma} = F_*\gamma$ on $B(0, 2) \setminus B(0, 1)$, and $\hat{g} = \text{metric}$ associated to $\hat{\gamma}$. In spherical coordinates $(r, \phi, \theta) \rightarrow (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$, we have

$$\hat{\gamma} = \begin{pmatrix} 2(r-1)^2 \sin \theta & 0 & 0 \\ 0 & 2 \sin \theta & 0 \\ 0 & 0 & 2(\sin \theta)^{-1} \end{pmatrix}.$$

The idea is that the origin is invisible. The ball in physical space with the blue fish inside corresponds to the origin in the virtual space. So the blue fish will be invisible. The precise result is in the following.

Theorem ([15]). *Let $\tilde{\gamma}$ (resp. \tilde{g}) be the conductivity (resp. metric) in $B(0, 2)$ such that $\tilde{\gamma} = \hat{\gamma}$ (resp. $\tilde{g} = \hat{g}$) on $B(0, 2) \setminus B(0, 1)$ and arbitrarily positive definite on $B(0, 1)$. Then we have*

$$\Lambda_{\tilde{\gamma}} = \Lambda_{\gamma} \quad (\text{resp. } \Lambda_{\tilde{g}} = \Lambda_g).$$

Quasilinear conductivities

Recently there has been progress for Calderón's problem in the case that the conductivity depends on the potential and the current, that is on the solution and gradient of the potential. We describe first the general problem when the conductivity depends on the potential and the gradient. This leads to an inverse problem for a quasilinear elliptic equation.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with C^∞ boundary. In [6], the authors considered the boundary value problem

$$\begin{cases} \nabla \cdot (\gamma(x, u, \nabla u) \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Here we assume that the conductivity function $\gamma : \overline{\Omega} \times \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}$ satisfies the following two assumptions:

(H1) $0 < \gamma(\cdot, 0, 0) \in C^\infty(\overline{\Omega})$,

(H2) The map $\mathbb{C} \times \mathbb{C}^n \ni (\rho, \mu) \rightarrow \gamma(\cdot, \rho, \mu)$ is holomorphic with values in the Hölder space $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

One can prove that under the assumptions (H1) and (H2), there exists $\delta > 0$ such that given any

$$f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\},$$

the problem has a unique solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|u\|_{C^{2,\alpha}(\overline{\Omega})} < C\delta$ with some constant $C > 0$. We define the associated DN map Λ_γ as

$$\Lambda_\gamma : f \mapsto (\gamma(x, u, \nabla u) \partial_\nu u)|_{\partial\Omega},$$

where $f \in B_\delta(\partial\Omega)$ and ν is the unit outer normal to $\partial\Omega$. In [6], the authors proved the following result:

Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with C^∞ boundary. Assume that $\gamma_1, \gamma_2 : \overline{\Omega} \times \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}$ satisfy (H1) and (H2). Suppose that there holds:*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f), \quad \forall f \in B_\delta(\partial\Omega).$$

Then,

$$\gamma_1 = \gamma_2 \quad \text{in } \overline{\Omega} \times \mathbb{C} \times \mathbb{C}^n.$$

The Calderón Problem for Fractional Operators

Recently, equations involving non-local operators have attracted much attention. A typical non-local operator is the fractional Laplacian $(-\Delta)^s$ given by

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u(\xi))(x),$$

where \mathcal{F} is the Fourier transform. The equivalent singular integral definition is

$$(-\Delta)^s u(x) := c_{n,s} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.$$

These kinds of operators have been introduced to describe nonlocal interactions. They are used to model anomalous diffusion and random processes with jumps in probability theory, in physics, finance, and biology. $(-\Delta)^s$ also appears in the formulation of fractional quantum mechanics. See [24] for details.

Inverse problems associated with fractional operators have been studied starting with the article [13]. Instead of the Dirichlet problem studied in the classical Calderón problem, the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \quad u|_{\Omega_e} = g$$

was considered in [13], where Ω is a bounded Lipschitz domain in \mathbb{R}^n , $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$, $n \geq 2$, $0 < s < 1$, $q \in L^\infty(\Omega)$. The inverse problem is whether one can determine the potential q in Ω from exterior partial measurements of the DN map

$$\Lambda_q : g \rightarrow (-\Delta)^s u|_{\Omega_e}.$$

It turns out that for $g \in C_c^\infty(W)$ where $W \subset \Omega_e$ is open, the knowledge of $\Lambda_q g|_W$ is equivalent to the knowledge of $\mathcal{N}_s u|_W$ where the nonlocal Neumann boundary operator \mathcal{N}_s is defined by

$$\mathcal{N}_s u(x) := c_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \Omega_e.$$

See Appendix in [13] for details for this equivalence as well as a probabilistic interpretation of Λ_q .

The following theorem is the fundamental uniqueness result proven in [13].

Theorem. *Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy that 0 is not an eigenvalue of the associated exterior Dirichlet problems. Let $W_1, W_2 \subset \Omega_e$ be open. If*

$$\Lambda_{q_1} g|_{W_2} = \Lambda_{q_2} g|_{W_2}, \quad g \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω .

The fractional Calderón Problem has several remarkable features when compared with its classical counterpart. Note that in the classical Calderón Problem, we treat the cases $n \geq 3$ and $n = 2$ separately while we deal with the fractional problem for all $n \geq 2$ in a uniform way. Although we still need to use the integral identity

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})g_1, g_2 \rangle = \int_{\Omega} (q_1 - q_2)u_1 u_2,$$

where u_j corresponds to the exterior data g_j ($j = 1, 2$), no complex geometrical solutions will be needed. Instead we use the following unique continuation property of the fractional Laplacian and the associated Runge approximation property to prove the uniqueness theorem, which is a strong partial data result. Both properties are typical non-local phenomena.

Theorem. Let $0 < s < 1$. Suppose $u \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Let $W \subset \mathbb{R}^n$ be open and non-empty. If

$$(-\Delta)^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ in \mathbb{R}^n .

Theorem. Let $q \in L^\infty(\Omega)$ satisfy that 0 is not an eigenvalue of the exterior Dirichlet problem. Let $W \subset \Omega_e$ be open. Then

$$S := \{u_g|_\Omega : g \in C_c^\infty(W)\}$$

is dense in $L^2(\Omega)$ where u_g is the solution corresponding to the exterior data g .

The anisotropic fractional Calderón problem (in the same conformal class) has been solved in [12]. The corresponding problem for the local Calderón problem is still open.

4 Travel Time Tomography

Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

The question of determining the sound speed or index of refraction of a medium by measuring the first arrival times of waves arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. This inverse problem was considered by Herglotz [18] in 1905 assuming the index of refraction (which is inverse proportional to the speed) depends only on the radius. Moreover, assume that the sound speed $c(r)$ satisfies $\frac{d}{dr} \left(\frac{r}{c(r)} \right) > 0$, where $r = |x|$. Then he solved the inverse problem of recovering the radial sound speed from the travel times.

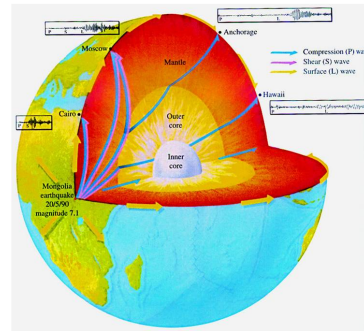
A more realistic model is to assume that it depends on position, the case of an heterogeneous medium. The travel time tomography problem can be formulated mathematically as determining a Riemannian metric on a bounded domain (the Earth) given by $ds^2 = \frac{1}{c^2(x)} dx^2$, where c is a positive function, from the length of geodesics (travel times) joining points in the boundary.

More recently it has been realized, by measuring the travel times of seismic waves, that the inner core of the Earth exhibits anisotropic behavior, that is, the speed of waves depends also on direction there with the fast direction parallel to the Earth's spin axis. Given the complications presented by modeling the Earth as an anisotropic elastic medium, we consider a simpler model of anisotropy, namely that the wave speed is given by a symmetric, positive definite matrix $g = (g_{ij})(x)$, that is, a Riemannian metric in mathematical terms. The problem is to determine the metric from the lengths of geodesics joining points in the boundary (the surface of the Earth in the motivating example). Other applications of travel time tomography are to imaging the Sun's interior [21], to medical imaging [33], and to ocean acoustics [28] to name a few.

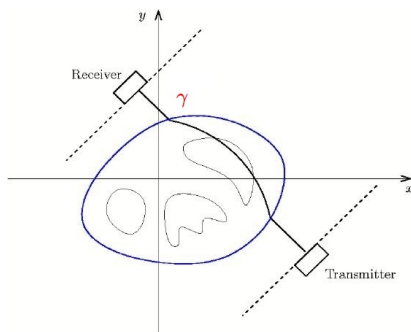
A general and geometric formulation of the travel time tomography problem is the question of whether given a compact Riemannian manifold with boundary one can determine the Riemannian metric in the interior knowing the lengths of geodesics joining points on the boundary, i.e. the boundary distance function. This is a problem that also appears naturally in rigidity questions in Riemannian geometry and it is known as the *boundary rigidity problem*.



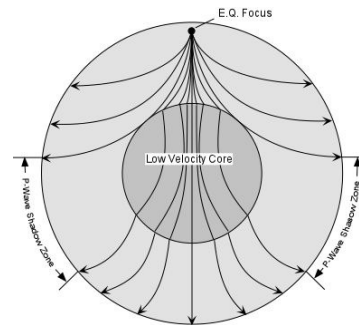
G. Herglotz



Mongolia earthquake (1990) from Wikipedia



Ultrasound transmission tomography



Earthquakes

Now we formulate this problem more precisely. Let (M, g) be a compact Riemannian manifold with boundary, where in local coordinates $g = (g_{ij})$ is a positive definite symmetric metric. The distance function is defined by

$$d_g(x, y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma),$$

where $L(\sigma)$ is the length of a piecewise smooth curve σ joining x and y . In local coordinates,

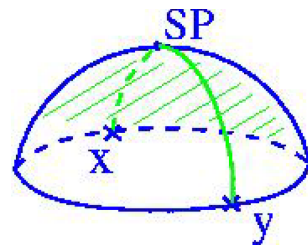
$$L(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt.$$

The inverse problem is to determine g knowing $d_g(x, y)$ for all $x, y \in \partial M$.

It is easy to check that if $\psi : M \rightarrow M$ is a diffeomorphism with $\psi|_{\partial M} = \text{Identity}$, then $d_{\psi^*g} = d_g$, where $\psi^*g = (D\psi \circ g \circ (D\psi)^T) \circ \psi$.

The boundary rigidity problem is whether this is the only obstruction to determine g from d_g . In general the answer is no. For example, if we increase slightly the standard metric on the southern hemisphere near the south pole, then the length minimizing curve joining the point x to the point y will not go to the south pole. We cannot get any information about the metric near the south pole from the distance function.

Definition. We say (M, g) is boundary rigid if (M, \tilde{g}) satisfies $d_{\tilde{g}} = d_g$. Then $\exists \psi : M \rightarrow M$ a diffeomorphism with $\psi|_{\partial M} = \text{Identity}$, so that $\tilde{g} = \psi^*g$.



From the example, we know that we need an a-priori condition for (M, g) to be boundary rigid. One such condition is that (M, g) is simple.

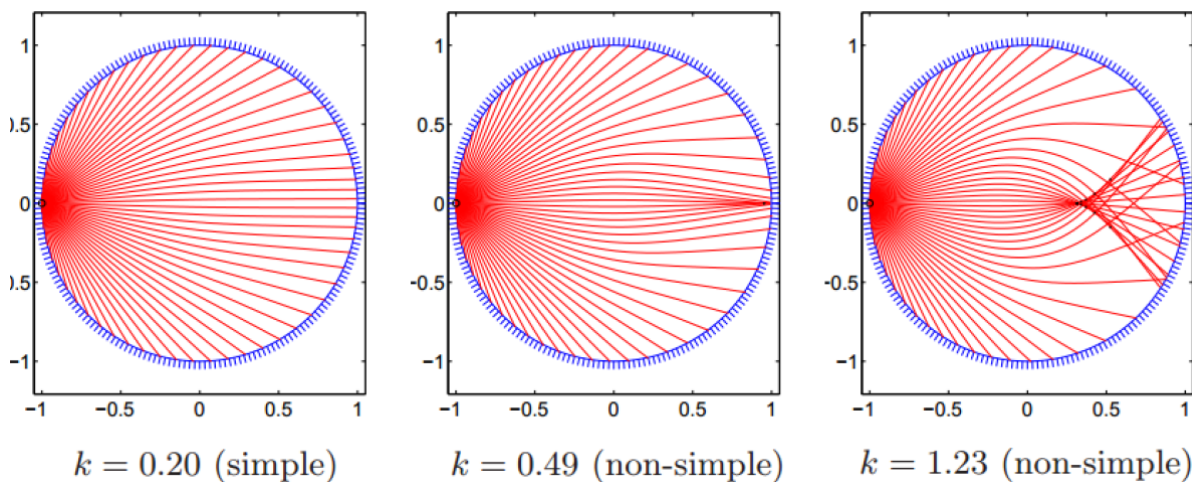
Definition. We say (M, g) is simple if given two points $x, y \in \partial M$, $\exists!$ minimizing geodesic joining x and y and ∂M is strictly convex.

Michel's Conjecture: If (M, g) is simple then (M, g) is boundary rigid, that is d_g determines g up to the natural obstruction.

This conjecture was posed by R. Michel in 1981, see [29].

We remark that the Herglotz's condition for radius sound speed is not the same as simplicity, see the examples below in [11], where

$$g_k(r) = \exp\left(k \exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \quad 0 \leq r \leq 1, \quad \sigma \text{ fixed.}$$

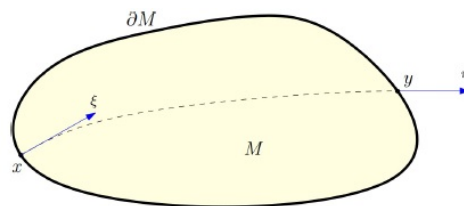


Michel's conjecture in two dimensions has been proved in [31]. Progress in dimension three or higher can be found in [37]. We discuss later in this article the recent progress obtained by studying the scattering relation.

The Scattering Relation: The distance function d_g only measures first arrival times of waves. In order to determine the metric without assuming simplicity, we need to look at behavior of all geodesics. This is called the scattering relation, which is defined by

$$\alpha_g(x, \xi) = (y, \eta), \quad \text{with } \|\xi\|_g = \|\eta\|_g = 1,$$

where (x, ξ) is the point and the direction of entrance of the geodesic and (y, η) is the point and direction of exit.

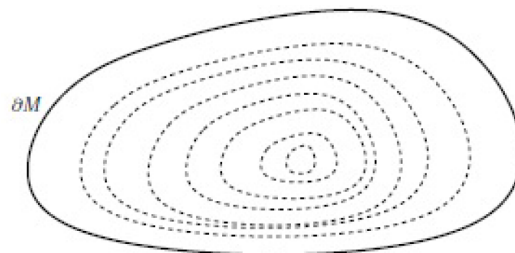


The lens rigidity problem is whether the scattering relation plus the length of the geodesic joining two points of the boundary, determines the Riemannian metric inside up to isometry. The lens rigidity problem and the boundary rigidity problems are equivalent for simple manifolds. The natural conjecture is that for non-trapping manifolds (all geodesics have finite length) the scattering relation determines the metric up to isometry.

Conjecture: Every non-trapping Riemannian manifold with boundary is lens rigid.

This conjecture has been proven in [34] assuming the manifold satisfies the foliation condition that we define below.

Definition. Let (M, g) be a compact Riemannian manifold with boundary. We say that M satisfies the foliation condition by strictly convex hypersurfaces if M is equipped with a smooth function $\rho : M \rightarrow [0, \infty)$ which level sets $\Sigma_t = \rho^{-1}(t)$, $t < T$ with some $T > 0$ are strictly convex viewed from $\rho^{-1}((0, t))$ for g , $d\rho$ is non-zero on these level sets, and $\Sigma_0 = \partial M$ and $M \setminus \cup_{t \in [0, T)} \Sigma_t$ has empty interior.



The foliation condition assumed in [34] is an analog of the Herglotz's condition $\frac{\partial}{\partial r} \frac{r}{c(r)} > 0$, with $\frac{\partial}{\partial r} = \frac{x}{|x|} \cdot \partial_x$ the radial derivative, proposed by Herglotz [18] for an isotropic radial sound speed $c(r)$. In this case the geodesic spheres are strictly convex.

In fact, we have the following proposition, which extends the Herglotz's result to not necessarily radial speeds $c(x)$. Let $B(0, R)$, $R > 0$ be the ball in \mathbb{R}^n , $n \geq 3$ centered at the origin with radius $R > 0$. Let $0 < c(x)$ be smooth in $B(0, R)$.

Proposition. *The Herglotz condition for $0 < r = |x| \leq R$ is equivalent to the condition that the Euclidean spheres $S_r = \{|x| = r\}$ are strictly convex in the metric $c^{-2}dx^2$ for $0 < r \leq R$.*

A special important case arises when there exists a strictly convex function, which may have a critical point x_0 in M . (If so, it is unique.) This condition was extensively studied. In particular it has been shown that such a function exists if any one of the following conditions holds:

- (1) The sectional curvature is non-negative.
- (2) M is simply connected with no focal points.
- (3) M is simply connected and the curvature is non-positive.

Note that the foliation condition allows for conjugate points. The main result in [34] is the following theorem.

Theorem. *Suppose that (M, g) is a compact n -dimensional Riemannian manifold, $n \geq 3$, with strictly convex boundary, and x is a smooth function with non-vanishing differential whose level sets are strictly concave from the superlevel sets, and $\{x \geq 0\} \cap M \subset \partial M$. Suppose also that \hat{g} is a Riemannian metric on M and suppose that the lens data of g and \hat{g} are the same. Then there exists a diffeomorphism $\psi : M \rightarrow M$ fixing ∂M such that $g = \psi^*\hat{g}$.*

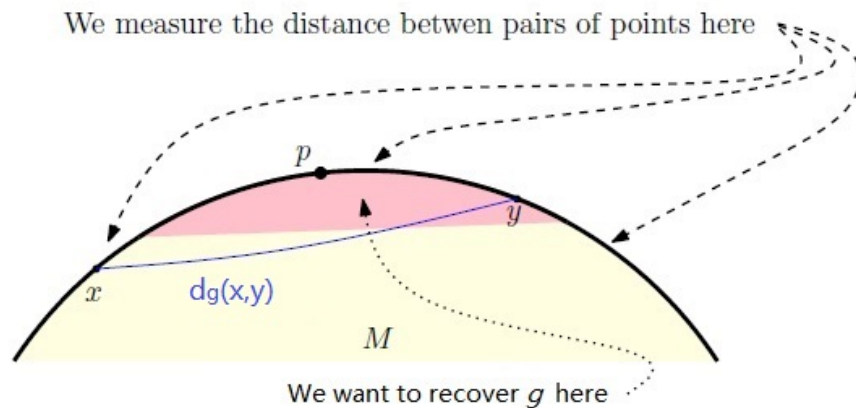
Corollary. *Simple manifolds satisfying the foliation condition are boundary rigid.*

Conjecture: Simple manifolds satisfy the foliation condition.

This theorem was proven by solving the boundary rigidity problem near the boundary and then matching in using a layer-stripping argument. The foliation condition is used to do the layer stripping, since we need the strictly convex point for the continuation of the process.

Boundary Rigidity with Partial Measurements

Let Γ be a non-empty open subset of the boundary of a compact Riemannian surface with boundary. We consider the problem of determining the metric near Γ . This problem has been solved in dimensions three and higher in [34].



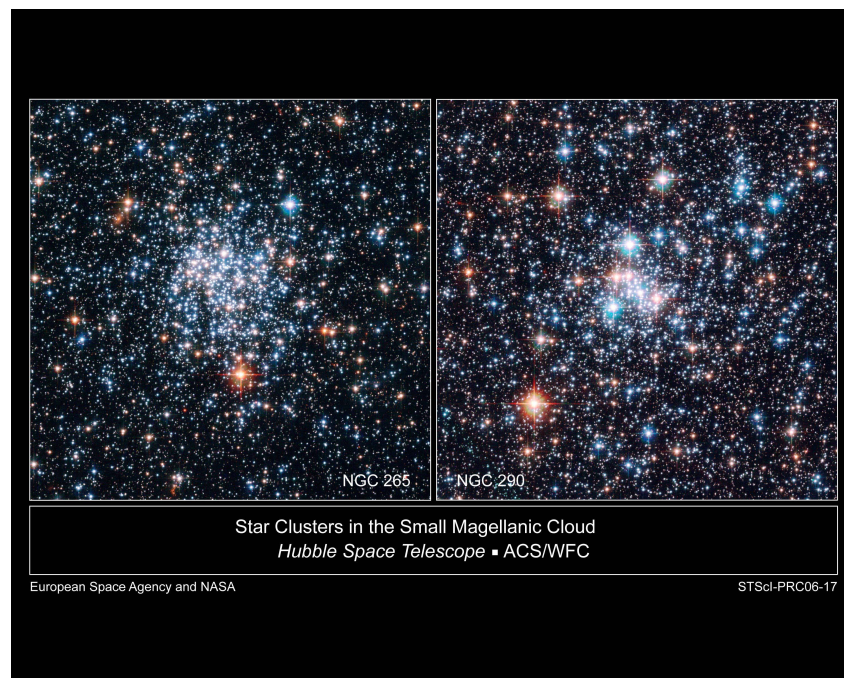
Let (M, g) be compact with smooth boundary. Linearizing $g \mapsto d_g$ in a fixed conformal class leads to the geodesic X-ray transform

$$If(x, \xi) = \int_0^{\tau(x, \xi)} f(\gamma(t, x, \xi)) dt$$

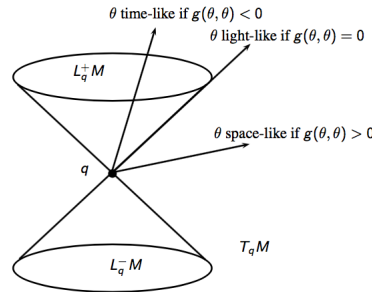
where $x \in \partial M$ and $\xi \in S_x M = \{\xi \in T_x M; |\xi| = 1\}$. Here $\gamma(t, x, \xi)$ is the geodesic starting from point x in direction ξ , and $\tau(x, \xi)$ is the time when γ exits M . We assume that (M, g) is nontrapping, i.e. τ is always finite. In the case the metric g is Euclidean, the geodesic X-ray transform is the standard X-ray transform. The local problem for the X-ray transform was solved in [38]. In this article, an inversion formula was shown for the local geodesic transform, meaning considering $If(\gamma) = \int f(\gamma(s)) ds$ known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) “almost tangentially”. It is proven in [38] that those integrals determine f near p uniquely. A reconstruction formula is given. In the case of the Euclidean metric, this is known as the Helgason support theorem. This was extended by H. Zhou for more general curves and non-vanishing weights.

5 Inverse Problems in General Relativity and Nonlinear Hyperbolic Equations

The motivation for the first problem considered in this section is whether we can determine the structure of spacetime when we see light coming from many point sources varying in time. We can also observe gravitational waves.

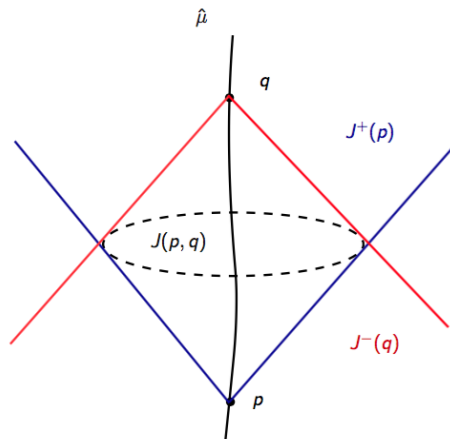


Let (M, g) be a 1 + 3 dimensional time oriented Lorentzian manifold. The signature of g is $(-, +, +, +)$. An example is Minkowski space-time (\mathbb{R}^4, g_m) , $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.



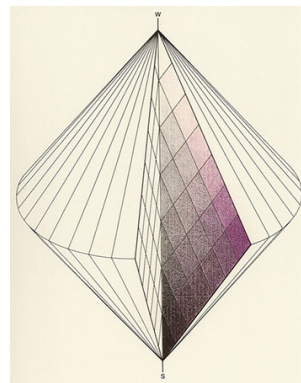
In the picture above $L_q^\pm M$ is the set of future (past) pointing light like vectors at q . Casual vectors are the collection of time-like and light-like vectors. A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).

Let $\hat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.



The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$. The causal future of $p \in M$ is $J^+(p) = \{q \in M : p < q\}$. $J(p, q) = J^+(p) \cap J^-(q)$, $I(p, q) = I^+(p) \cap I^-(q)$.

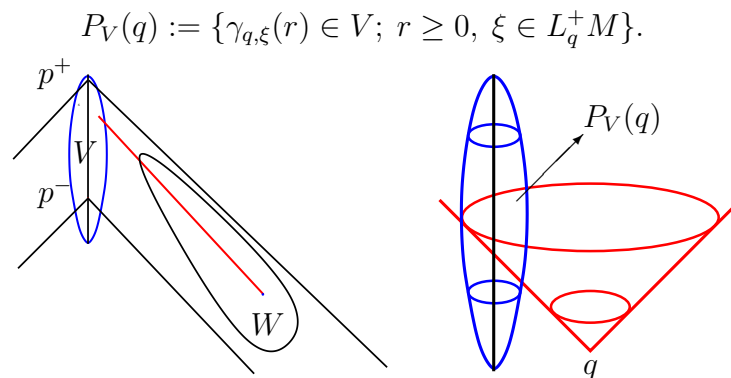
A Lorentzian manifold (M, g) is globally hyperbolic if there is no closed causal paths in M and for any $p, q \in M$ and $p < q$, the set $J(p, q)$ is compact.



Then hyperbolic equations are well-posed on (M, g) . Also, (M, g) is isometric to the product manifold $\mathbb{R} \times N$ with $g = -\beta(t, y)dt^2 + \kappa(t, y)$. Here $\beta : \mathbb{R} \times N \rightarrow \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t .

We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M . Let $\mu = \mu([-1, 1]) \subset M$ be time-like geodesics containing p^- and p^+ . We consider observations in a neighborhood $V \subset M$ of μ . Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

Definition. *The light observation set for $q \in W$ is*



Definition. *The earliest light observation set of $q \in M$ in V is*

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{there is no } y \in \mathcal{P}_V(q) \text{ and future pointing time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts ([10]).

The light observation set is conformally invariant since it only uses light-like geodesics. The next result proven in [23] states that from the earliest light observation set we can recover the metric up to conformal factor.

Theorem. *Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\hat{\mu}([-1, 1]) \subset M, p^\pm = \hat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\hat{\mu}([-1, 1])$ and $W \subset M$ be a relatively compact set. Assume that we know $\mathcal{E}_V(W)$. Then we can determine the topological structure, the differential structure, and the conformal structure of W , up to diffeomorphism.*

Active Measurements

In the paper [22] a new method was introduced to solve inverse problems for hyperbolic systems where the leading order terms depend non-linearly on the solution. This paper considers an inverse problem for Einstein's equations coupled with scalar fields.

The method developed in [22] utilizes the non-linearity as a tool and it enables us to solve inverse problems for non-linear equations even in cases where the inverse problem for the corresponding linear system remains open. Indeed, the existing uniqueness results for linear hyperbolic equations require restrictive geometric assumptions such as stationarity, whereas the result of the paper is applicable to any globally hyperbolic spacetime. We review the existing literature on inverse problems in detail.

In physical terms, we study the question: Can an observer determine the structure of the surrounding spacetime by doing measurements near its world line? The conservation law for Einstein's equations dictates, roughly speaking, that any source in the equation must take energy from some fields in order to increase energy in other fields. The scalar fields are included in the model that we consider to facilitate this. They correspond to spin zero particles.

We now give the mathematical formulation of the problem. Let M be a smooth $1 + 3$ -dimensional manifold. Einstein's equations for a Lorentzian metric g on M are

$$\text{Ein}(g) = T,$$

where $\text{Ein}_{jk}(g) = \text{Ric}_{jk}(g) - \frac{1}{2}(g^{pq} \text{Ric}_{pq}(g))g_{jk}$, $\text{Ric}(g)$ is the Ricci tensor. T denotes the stress-energy tensor. In vacuum, $T = 0$. In wave-map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law, $\nabla_p(g^{pj}T_{jk}) = 0$.

One can not do measurements in vacuum, so matter fields need to be added. We can consider the coupled Einstein and scalar field equations with sources. The measurement model we consider is

$$\begin{aligned} \text{Ein}(g) = T, \quad T &= \mathbb{T}(g, \phi) + \mathcal{F}^1, \quad \text{on } (-\infty, T) \times N, \\ \square_g \phi_l - m^2 \phi_l &= \mathcal{F}_2^\ell, \quad l = 1, 2, \dots, L. \\ g|_{t < 0} &= \hat{g}, \quad \phi|_{t < 0} = \hat{\phi}. \end{aligned}$$

Here \hat{g} and $\hat{\phi}$ are C^∞ -smooth and satisfy the same equation with the zero sources and

$$T_{jk}(g, \phi) = \sum_{\ell=1}^L \partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

$$\nabla_p(g^{pj}T_{jk}) = 0, \quad k = 1, 2, 3, 4.$$

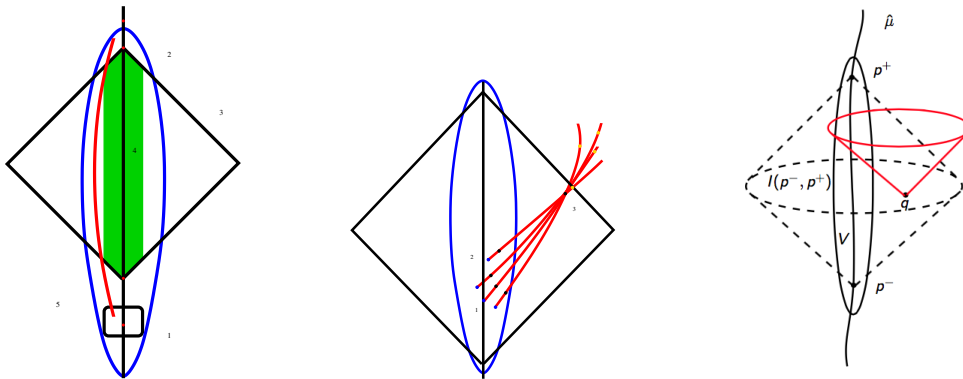
Let $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2) = (\mathcal{F}^1, \mathcal{F}_1^2, \dots, \mathcal{F}_L^2)$ model a source in the measurements. Let $V_{\hat{g}} \subset M$ be a neighborhood of the timelike geodesic μ and $p^-, p^+ \in \mu$.

Theorem. ([23]) *Let*

$$\mathcal{D} = \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g})\},$$

where g and ϕ satisfy Einstein equations with a source $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, $\text{supp}(\mathcal{F}) \subset V_g$, and $\nabla_j(\mathbf{T}^{jk}(g, \phi) + \mathcal{F}_1^{jk}) = 0$. The data set \mathcal{D} determines uniquely the conformal type of casual diamond $(J^+(p^-) \cap J^-(p^+), \hat{g})$.

The proof of the theorem involves the interaction of four distorted plane waves, whose normals are drawn below in red intersecting at the point q inside the casual diamond. The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



See the simulation in

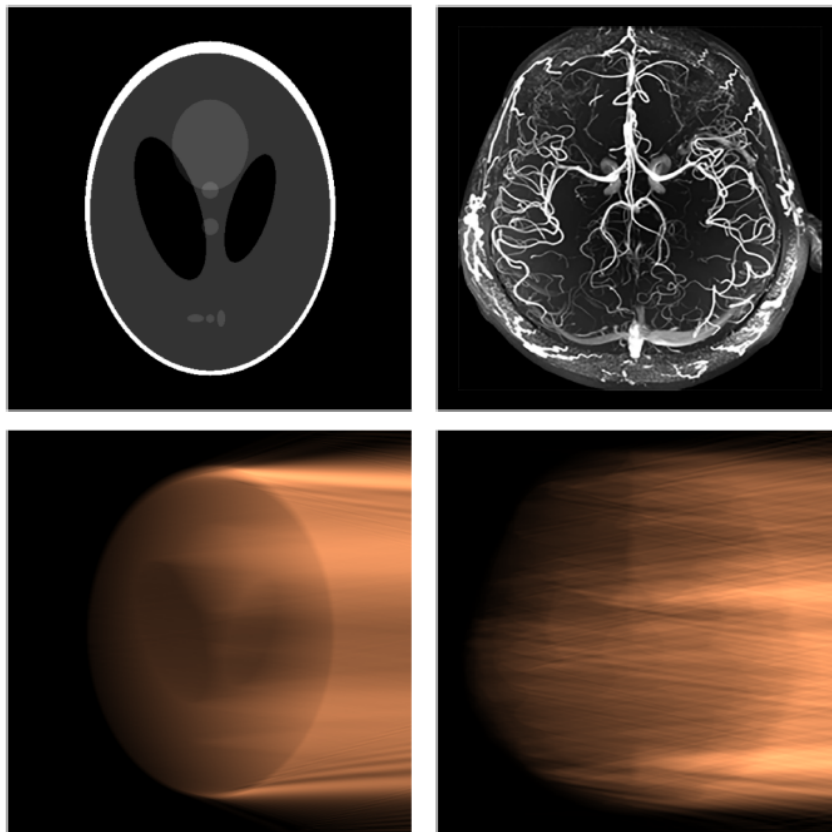
<https://wiki.helsinki.fi/display/mathstatHenkilokunta/video?preview=/244755256/244762293/talkmovie6.mp4>.

Nonlinear Acoustics

The methods used in [22] have been used to solve inverse problems in several nonlinear hyperbolic equations, including nonlinear elastic wave equations, see [39]. In this article, we consider applications to nonlinear acoustic wave equations.

We briefly summarize the main result of [2]. The motivation is *tissue harmonic imaging*. In this method, higher order frequency waves produced by nonlinear ultrasound waves are used to produce images that improve the images obtained by standard ultrasound, see [1]:

Nonlinear ultrasound waves play an increasingly important role in diagnostic and therapeutic medicine. The motivation is the portability of ultrasound-based technologies which makes them ideal for monitoring patients in the operating room. The main goal of [2] is to contribute to the mathematical understanding of quantitative nonlinear ultrasound imaging. The mathematical question is whether boundary measurements of the ultrasound field can uniquely determine the coefficient of nonlinearity in the wave equation. The starting point is a lossless nonlinear wave equation of Westervelt type governing the propagation of waves in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$.



True profiles for the Shepp-Logan phantom (top left) and brain vasculature (top right). The amplitude of the numerical solutions v for $\theta = 0$ (bottom row) for $L/\lambda = 100$ where λ is the wavelength of the probing field.

We now describe the model of wave propagation in nonlinear media that was used in [2] and the corresponding inverse problem.

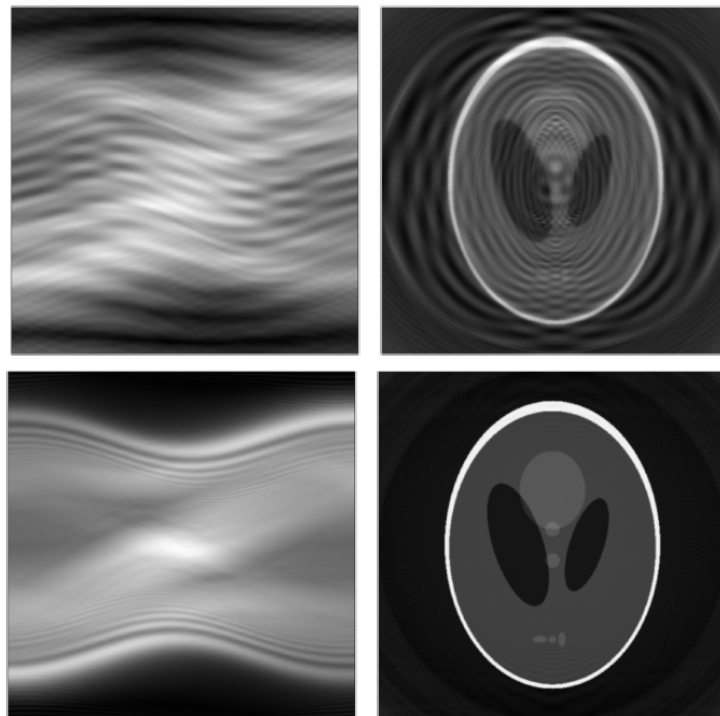
Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Consider the following quasilinear wave equation for the pressure $p(x, t)$ in [9].

$$\begin{aligned} \partial_t^2 p(t, x) - c(x)\Delta p(t, x) - \beta(x)\partial_t^2 p^2(t, x) &= 0, & \text{in } (0, T) \times \Omega, \\ p(t, x) &= f & \text{on } (0, T) \times \partial\Omega, \\ p(0, x) = \partial_t p(0, x) &= 0. \end{aligned}$$

Here $c(x)$ denotes the sound speed (assumed to be known). We are interested in determining the coefficient of the non-linearity $\beta(x)$ which is a measure of acoustic nonlinearity. We assume that the sound speed depends on the space variable and is known. Consider the Dirichlet-to-Neumann map Λ defined as

$$\Lambda f = \partial_\nu p|_{(0, T) \times \partial\Omega}.$$

The main result of [2] is that we can uniquely recover β from Λ . Numerical simulations in [2] are given below. The proof of this result involves the interaction of two distorted plane waves.



Synthetic data using the forward model and reconstructed profiles using the inversion algorithm for probing ultrasound waves of frequencies such that $L/\lambda = 10$ (top row) and $L/\lambda = 100$ (bottom row) where L is the size of the image and λ is the wavelength.

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Time's Arrow

Scientific anniversaries

1971. The Association for Women in Mathematics was founded.

1991. In August Paul Ginsparg created what is now known as the arxiv at the Los Alamos National Laboratory, and Tim Berners-Lee published the first website at CERN.

Recent personal celebrations

Yakov Sinai [was honored upon attaining emeritus status at Princeton University](#).

Lost luminaries

Maria da Conceição Vieira de Carvalho, 28 June, 2021 ([English, português](#)).

[Robin Hudson](#), 12 January, 2021.

[Victor Isakov](#), 14 May, 2021.

[Roger Lewis](#), 8 June, 2021.

[Andrew Majda](#), 12 March, 2021. See also [a remembrance by Di Qi](#).

[Martin Schechter](#), 7 June, 2021.

Readers are encouraged to send items for "Time's Arrow" to bulletin@iamp.org.

News from the IAMP Executive Committee

General Assembly, Geneva, August 2-7, 2021.

The General Assembly of the IAMP will be held during the ICMP on August 4, 2021. It starts at 16:00 and ends at 17:50 Swiss time, i.e., CET.

You can attend the meeting in person or online, but you have to register for the ICMP (online registration if you plan to participate online). The Assembly is incorporated on the online platform and there is no access for people that are not registered.

As always, there will be reports by the President and the Treasurer, and a presentation of the bids for the 2024 ICMP followed by a general discussion.

New individual members

IAMP welcomes the following new members

1. PROFESSOR NALINI ANANTHARAMAN, University of Strasbourg, Strasbourg, France
2. DR. TED CLARKE, Christian Brothers University, Memphis, TN, USA
3. PROFESSOR SEMYON KLEVTSOV, University of Strasbourg, Strasbourg, France
4. MR. CHRISTOPH MINZ, University of York, York, UK
5. DR. JENIFER STEFFI, St. Xavier's College, Palayamkottai, Tamil Nadu, India
6. PROFESSOR CLÉMENT TAUBER, University of Strasbourg, Strasbourg, France
7. DR. VEDRAN SOHINGER, University of Warwick, Warwick, UK
8. DR. AMIR ABBASS VARSHOVI, University of Isfahan, Isfahan, Iran

Recent conference announcements

Quantum Trajectories Fall School

October 18-22, 2021, Toulouse, France.

Call from the Centre for Interdisciplinary Research (ZiF), Bielefeld University, Germany.

The ZiF, Bielefeld's international institute for advanced study, has issued an annual call for doing interdisciplinary research in a group during 2023/24 for 5 or 10 months. Applications combining mathematics and physics (and other sciences) are highly welcome.

For further information please consult this page:

https://www.uni-bielefeld.de/ZiF/Aktuell/Call_for_Project_Proposals.pdf.

Open positions

For an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

Michael Loss (IAMP Secretary)

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