IAMP News Bulletin
October 2018

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

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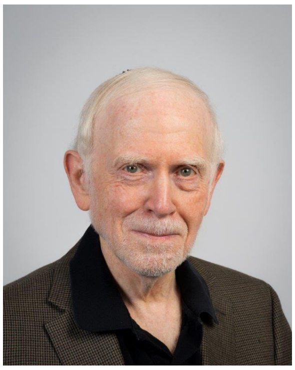
Prizes from the IAMP

On July 23, at the 2018 International Congress in Montreal, Canada, the International Association of Mathematical Physics (IAMP) awarded the 2018 Henri Poincaré Prizes for mathematical physics to Michael Aizenman, Princeton University; Percy Deift, New York University; and Giovanni Gallavotti, Università di Roma La Sapienza.

Michael Aizenman was honored "for his seminal contributions to quantum field theory, statistical mechanics, and disordered systems in which he pioneered innovative techniques that demonstrate the beautiful and effective interplay between physical ideas, mathematical analysis, geometric concepts, and probability theory."



Percy Deift was honored "for his seminal contributions to Schrödinger operators, inverse scattering theory, nonlinear waves, asymptotic analysis of Fredholm and Toeplitz determinants, universality in random matrix theory, and his deep analysis of integrable models."

Giovanni Gallavotti was honored "for his outstanding contributions to equilibrium and non-equilibrium statistical mechanics, quantum field theory, classical mechanics, and chaotic systems, including, in particular, the renormalization theory for interacting fermionic systems and the fluctuation relation for the large deviation functional of entropy production."

The Henri Poincaré Prize, which is sponsored by the Daniel Iagolnitzer Foundation, recognizes outstanding contributions that lay the groundwork for novel developments in mathematical physics. It also recognizes and supports young people of exceptional promise who have already made outstanding contributions to the field. The prize is awarded every three years at the International Congress on Mathematical Physics.

IAMP awarded the 2018 Early Career Award to *Semyon Dyatlov* (UC Berkeley / Massachusetts Institute of Technology) "for the introduction and the proof of the fractal uncertainty principle, which has important applications to quantum chaos and to observability and control of quantum systems."

The Early Career Award is given in recognition of a single achievement in Mathematical Physics and is reserved for scientists whose age is less than 35. It is sponsored by Springer.

For prior winners, selection committee members and laudations, see http://www.iamp.org.

In Noether's Words: Invariant Variational Problems

by YVETTE KOSMANN-SCHWARZBACH (Paris)

To celebrate the centenary of the publication of Invariante Variationsprobleme*, Noether's fundamental article on symmetries and conservation laws of variational problems, I recall its origin in a problem posed by the general theory of relativity.*

Why is Noether so famous?

Emmy Noether's father was the mathematician Max Noether, one of the foremost specialists in algebraic geometry at the end of the nineteenth century. For decades, dictionaries would have an entry for Noether (Emmy) such as "Emmy Noether, daughter of the mathematician Max Noether, born in Erlangen (Bavaria, Germany) in 1882, died in Bryn Mawr, near Philadelphia (Pennsylvania) in the United States in 1935, ... " while nowadays, an entry for Noether (Max) is usually, "Max Noether (1844–1921), father of the mathematician Emmy Noether, ... "

Why is Emmy Noether so famous? Because she was one of the greatest algebraists of the first half of the twentieth century, the author of fundamental articles on the structure of rings and the representations of groups. Furthermore, she wrote *Invariante Variationsprobleme* in 1918 and this article made her famous in the world of mathematical physicists, but only long after it was published in the *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse.* Noether herself never referred to her paper except in a letter to Einstein in 1926, by which time she was no longer sure of its date, "1918 or 19." In fact, this article was a logical extension of her study of the theory of invariants, but it remained unrelated to her later work in abstract algebra.

What did she write in this 23-page paper on invariant variational problems? Why is it so important? How did she become famous among the students and the scientists studying classical and quantum mechanics, quantum field theory, the theory of general relativity and many other domains of science? The centenary of her article is an occasion to reflect on these questions.

Two fundamental theorems

First, let me dispel the fantasies that have been written about Noether's work: No solar orbits and no bicycles are mentioned in her 1918 paper, contrary to what has been claimed in some popular articles about her and symmetries. In fact, she was not a physicist, but the paper was suggested by a problem in general relativity that Felix Klein (1849–1925) and David Hilbert (1862–1943), professors at the university of Göttingen, were trying to explain and about which they corresponded with Einstein who was working on the field equations of gravitation in Berlin. He had visited Göttingen for a week at the end of June 1915 and lectured there on what became known as the Entwurf (preliminary version) of general relativity. Upon his return to Berlin, he had written to two friends that he "had the pleasure of completely convincing the mathematicians," that he had been "understood to the last detail," and that he was "delighted with Hilbert"[1]. Because the problem of the conservation of energy in general relativity was related to the invariance property of the field equations, and because Noether had written her thesis in Erlangen on invariant theory, under the direction of Paul Gordan (1837–1912)[2], a colleague of her father, Klein and Hilbert invited her to come to Göttingen to help them in their search for an interpretation of the law of conservation of energy in general relativity. That is the problem she was able to solve three years later.

Noether's article contains two fundamental theorems, not one[3]. In the first section, she states both theorems.

I. *If the integral I is invariant under a* [*group*] $\mathfrak{G}_\rho$, *then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences—and conversely, this implies the invariance of I under a* [*group*] $\mathfrak{G}_\rho$. *The theorem remains valid in the limiting case of an infinite number of parameters.*

II. *If the integral I is invariant under a* [*group*] $\mathfrak{G}_{\infty\rho}$, *depending on* [$\rho$] *arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.*

In more modern terms, the first theorem deals with the tight relationship between the symmetries of the action integral of a Lagrangian theory and conservation laws[4], such as those in electrodynamics that follow from the conformal invariance of Maxwell's equations[5]. Theorem I has applications in classical and quantum mechanics, while Theorem II deals with infinite-dimensional groups of symmetries, such as the group of diffeomorphisms of the space-time manifold of general relativity and the gauge groups of gauge theory. Theorem II shows that infinite-dimensional groups of symmetries imply identities among the field equations. One original trait of Noether's treatment is that in both cases, she proves a converse of her theorem.

[1] These expressions are excerpts from Einstein's correspondence published in *The Collected Papers of Albert Einstein*, John Stachel *et al.*, eds., Princeton University Press, vol. 8A, 1998.

[2] The "Clebsch-Gordan" coefficients in quantum mechanics bear the name of the mathematician Paul Gordan, Emmy Noether's Doktorvater, together with that of the physicist and mathematician Alfred Clebsch (1833–1872). Gordan should not be confused with the twentieth-century physicist Walter Gordon (1893–1939). The"Klein-Gordon equation" is named after Gordon and the physicist Oskar Klein (1894–1977) who, in turn, should not be confused with the mathematician Felix Klein.

[3] For the best account to this day of Noether's two theorems and their modern formulation, see Peter Olver, *Applications of Lie Groups to Differential Equations*, Springer, 1986, 2nd edition 1993. For an English translation of *Invariante Variationsprobleme* and an account of the origins and the posterity of Noether's two theorems, see Yvette Kosmann-Schwarzbach, *The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century*, translated by Bertram E. Schwarzbach, Springer, 2011, revised edition 2019.

[4] Conservation laws are also called continuity equations. In the case where time is the single independent variable, they express the constancy of "conserved quantities."

[5] The conformal invariance of Maxwell's equations was proved by Harry Bateman (1882–1946) and by Ebenezer Cunningham (1881–1977). Both were mathematicians and both published this result in 1910.

Theorem I: Conservation Laws

The proof of the first theorem is not particularly difficult: it is an application of the method of integration by parts. Noether considers an infinitesimal symmetry of a Lagrangian and derives the corresponding conservation law for the associated system of Euler–Lagrange equations – whose left-hand sides she called "the Lagrangian expressions," a result that subsumes many known properties. A correlation between invariances and the principles of conservation was already known to Joseph-Louis Lagrange (1736–1813), who had developed, in his *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies*[6] (1760), the calculus of variations, the principle of which Euler had introduced in 1744 in his *Methodus inveniendi lineas curvas, maximi minimive proprietate gaudentes*[7]. In the time-evolution of a system described by the Euler–Lagrange equations associated with a Lagrangian invariant under translations, the linear momentum is conserved, and similarly for the angular momentum for a Lagrangian invariant under rotations: these results can be found in Lagrange's treatise, *Mécanique analytique* of 1811. What is remarkable is the generality of Noether's approach. The infinitesimal invariance that she considered was much more general than invariance under ordinary vector fields. Half a century after she introduced the concept of vector fields whose components depend on the dependent variables as well as on their derivatives of arbitrarily high orders, these generalized symmetries were rediscovered by many authors, starting with H. H. Johnson in 1964, who called them "a new type of vector field" and, independently, by Robert Hermann; they were called Lie–Bäcklund transformations by Robert L. Anderson and Nail H. Ibragimov in 1979, although they can be found neither in the works of Sophus Lie (1842–1899) nor in those of A. V. Bäcklund (1845–1922), until Olver, in his book in 1986, calling 'generalized vector fields' what Noether had called 'infinitesimal transformations', stressed that they had been introduced by her in her article and showed the importance of this concept in the applications to differential equations, and Alexandre M. Vinogradov interpreted them as vector fields on the infinite jet bundle of a vector bundle whose base is the space of independent variables and whose sections are the fields whose components are the dependent variables. This concept is a prime ingredient in the theory of completely integrable systems and soliton equations which developed after 1975.

Theorem II: Identities

Noether's second theorem was the key to an explanation of the problem posed by the conservation of energy in general relativity. She considered the case where a Lagrangian is invariant under ρ symmetries, each of which depends linearly on an arbitrary function and its derivatives up to a certain order. Such a symmetry is, in fact, defined by a vector-valued linear differential operator. From the invariance condition of the Lagrangian, she deduced ρ identities satisfied by the "Lagrangian expressions" (*i.e.*, the components of the Euler–Lagrange differential) and their partial derivatives. She obtained this result by remarking that, in order to express the invariance condition, the recourse to the formula of integration by parts that was sufficient to

[6] A new method that attempts to determine the maxima and minima of indefinite integral formulas.

[7] Method to determine the curves that have the property of being maximal or minimal.

derive her first theorem must be replaced by a consideration of the adjoint[8] of the differential operator defining the symmetry, which would be applied to a linear combination of the components of the Euler–Lagrange differential. That an identity among these components and their derivatives is satisfied then follows "by the classical rules of the calculus", *i.e.*, by the result now called the Du Bois-Reymond lemma, after the Swiss mathematician Paul Du Bois-Reymond (1831–1889), that permits concluding that if the integral of the product of a given function by an arbitrary function vanishes, then the given function vanishes.

How did this result help in the resolution of the question of energy conservation in general relativity ? In the case considered in her second theorem, Noether obtained what she called "improper divergence relations," *i.e.*, either a conservation law that is identically satisfied regardless of whether or not the field equations are satisfied (the conservation law holds "off-shell"), or a divergence relation for a quantity such that not only its divergence vanishes as a consequence of the Euler–Lagrange equations, but the quantity itself vanishes "on shell." This mathematical result enabled Noether to examine and vastly generalize an assertion made by Hilbert stating that, "in the case of general relativity and in that case only," there are no proper conservation laws. She observed that "the term relativity that is used in physics, should be replaced by invariance with respect to a group", and that the lack of proper conservation laws in general relativity is explained by the fact that the finite-dimensional Lie groups of invariance of the Lagrangian are in fact subgroups of the infinite-dimensional invariance group of the theory.

Noether's insight would apply to Hermann Weyl's theory published in *Elektron und Gravitation* in 1929, and to all later gauge theories.

Emmy Noether was a woman and a Jew

I shall not dwell on the fact that Noether was an outstanding mathematician many years before women began to enter various fields of research at the highest level. Yet this evidence should be stressed and makes her achievements ever more singular.

Why did she not die peacefully in Göttingen? She had certainly become famous enough, in her own time, had invitations from fellow mathematicians to visit the USSR and, in 1932, was invited to be a plenary speaker at the International Congress of Mathematicians, held every four years, although as a woman she was never named to a professorship in her own university. Why did she have to leave for the United States in 1933? This question does not need a long explanation. The Nazis had seized power in Germany and, as of 13 September 1933, as a Jew, she was excluded from the civil service. She found a position at Bryn Mawr College from where she could travel to the Institute of Advanced Study in Princeton, where she could talk to Einstein. She died after an operation two years after she had left Germany, her home country.

[8] The adjoint, D^*, of the linearization of a linear differential operator, D, satisfies $u\,(Dv) = (D^*u)\,v$ up to divergences. The concept of adjoint operator, due to Lagrange, was already classical in Noether's time: it figured prominently in the work of Lazarus Fuchs (1833–1902) and in the Sorbonne lectures of Vito Volterra (1860–1940), published in 1913.

Noether's fame

Tens of authors have re-discovered her first theorem or falsely claimed to have generalized it, and hundreds of mathematical physicists and relativists have applied either her first theorem or her second. The impact of her 1918 article on various fields of physics is undeniable. After 1970, genuine generalizations were discovered in both the geometric and the algebraic languages. Since then, her foray into a problem in general relativity, one of her many achievements, in fact became the principal one in the eyes of mathematical physicists. In this year of the centenary of Noether's "Invariant variational problems", we are allowed to think of her primarily as the author of this epoch-making article, while we remember that she was "also" - in addition so to speak - the creator of modern algebra.

Noether's landmark paper one hundred years later

by MARIA CLARA NUCCI (Paris)

1 Introduction

We honor the centennial of the publication of Noether's landmark paper "Invariante Variationsprobleme" with an overview of our research where the importance of Noether symmetries in classical and quantum mechanics has been demonstrated. We refer to the excellent book [12] for the historical background and thorough account of the developments of Noether's work in various fields, and to [29] for a modern mathematical formulation.

As tersely stated in [30]: *The First Noether Theorem establishes the connection between continuous variational symmetry groups and conservation laws of their associated Euler-Lagrange equations. The Second Noether Theorem deals with the case when the variational symmetry group is infinite-dimensional, depending on one or more arbitrary functions of the independent variables, e.g., the gauge symmetry groups arising in relativity and physical field theories.*

Here we are mainly concerned with Noether's First Theorem [2]. Therefore, what we call Noether's theorem below is actually Noether's First Theorem.

The variational problem most familiar to physicists is that of a Lagrangian of first-order, *i.e.*,

$$\mathcal{L} = \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)), \tag{1}$$

and its corresponding (Euler)-Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k}, \quad (k = 1, \dots, n). \tag{2}$$

If a Lie symmetry is represented by the following operator:

$$\Gamma = \xi(t, \mathbf{q})\partial_t + \sum_{k=1}^{n} \eta_k(t, \mathbf{q})\partial_{q_k}, \tag{3}$$

then a Noether symmetry has to satisfy the following relationship:

$$\mathcal{L}\frac{d\xi}{dt} + \Gamma_1(\mathcal{L}) = \frac{df}{dt}, \tag{4}$$

where $f = f(t, \mathbf{q})$ is a function to be determined, and

$$\Gamma_1 = \Gamma + \left(\frac{d\eta_k}{dt} - \dot{q}_k\frac{d\xi}{dt}\right)\partial_{\dot{q}_k}$$

is the first prolongation of Γ.

Thus, Noether's theorem yields the following first integral[9] of the Lagrangian equations:

$$\mathcal{I} = \xi\mathcal{L} + \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_k}(\eta_k - \dot{q}_k\xi) - f = \text{const}. \tag{5}$$

The key to find Noether symmetries is the boundary term f. In mechanics courses, students are usually taught very simple Noether symmetries of the natural Lagrangian (namely kinetic minus potential energy), e.g., translation in time, that yield f=constant. Indeed, one may have to deal with a very complicated f in order to find a Noether symmetry.

We begin by showing how easy it is to underestimate Noether's theorem even in the case of a simple quadratic ($n = 2$) Lagrangian [5], which indeed admits eight Noether symmetries [14]. Second, we describe a quantization method that preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation. Third, we show how Noether symmetries may be used to eliminate "ghosts" (*i.e.*, states with negative norm). Fourth, we search and find Lagrangians and consequently Noether symmetries for systems without Lagrangians. We conclude with some final remarks.

Missing Noether symmetries

The authors of [5] presented the following Lagrangian

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(q_1^2 + q_2^2) + tq_1, \quad (k, m = \text{const}), \tag{6}$$

and its corresponding Lagrangian equations

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{t}{m}, \qquad \ddot{q}_2 = -\frac{k}{m}q_2. \tag{7}$$

There three first integrals were determined, with the claim that only one was related to Noether symmetries. In [14], we derived eight Noether symmetries and corresponding conserved quantities.

System (7) is linear and in [14] we have shown that it admits a fifteen-dimensional Lie point-symmetry algebra, isomorphic to $sl(4, I\!R)$, and generated by the following fifteen operators:

$$\begin{aligned}
\Gamma_1 &= (kq_1 - t)\,\partial_{q_2} \\
\Gamma_2 &= -q_2 \sin\left(\sqrt{\frac{k}{m}}t\right)\partial_t + \frac{1}{k\sqrt{km}}\left(-kq_2(kq_1 - t)\cos\left(\sqrt{\frac{k}{m}}t\right)\right. \\
&\quad \left. -\sqrt{\frac{m}{k}}q_2 \sin\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_{q_1} - \sqrt{\frac{k}{m}}q_2^2 \cos\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}
\end{aligned}$$

[9] We use the terms *first integral*, *conserved quantity*, and *conservation law* interchangeably.

$$
\begin{aligned}
\Gamma_3 &= \left(q_1-\frac{t}{k}\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\partial_t+\frac{1}{k^2}(kq_1-t)\left(\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\left.-\sqrt{\frac{k}{m}}(kq_1-t)\sin\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_{q_1}-\frac{1}{\sqrt{km}}(kq_1-t)q_2\sin\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_4 &= -\left(q_1-\frac{t}{k}\right)\sin\left(\sqrt{\frac{k}{m}}t\right)\partial_t+\frac{1}{k^2\sqrt{mk}}\left(\left(-k(kq_1-t)^2-3m\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\left.+\sqrt{mk}(-kq_1+t)\sin\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_{q_1}-\frac{1}{\sqrt{mk}}(kq_1-t)q_2\cos\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_5 &= q_2\partial_{q_1}\\
\Gamma_6 &= \cos\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_7 &= -\sin\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_8 &= \left(1-2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_t+\frac{1}{k^2\sqrt{mk}}\left(-2k^2(kq_1-t)\sin\left(\sqrt{\frac{k}{m}}t\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\left.+k\sqrt{mk}\left(1-2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right)\right)\partial_{q_1}-2q_2\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_9 &= -2\cos\left(\sqrt{\frac{k}{m}}t\right)\sin\left(\sqrt{\frac{k}{m}}t\right)\partial_t+\frac{1}{k^2\sqrt{mk}}\left(-2k\sqrt{mk}\sin\left(\sqrt{\frac{k}{m}}t\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\left.-k^2(kq_1-t)\left(1-2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right)\right)\partial_{q_1}-q_2\sqrt{\frac{k}{m}}\left(1-2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_{q_2}\\
\Gamma_{10} &= k\partial_t+\partial_{q_1}\\
\Gamma_{11} &= q_2\partial_{q_2}\\
\Gamma_{12} &= q_2\cos\left(\sqrt{\frac{k}{m}}t\right)\partial_t+\frac{q_2}{k^2\sqrt{mk}}\left(k\sqrt{mk}\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\left.-k^2(kq_1-t)\sin\left(\sqrt{\frac{k}{m}}t\right)\right)\partial_{q_1}-q_2^2\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_2}\\
\Gamma_{13} &= -(kq_1-t)\partial_{q_1}\\
\Gamma_{14} &= \cos\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_1}\\
\Gamma_{15} &= \sin\left(\sqrt{\frac{k}{m}}t\right)\partial_{q_1}.
\end{aligned}
\tag{8}
$$

Then the Lagrangian (6) admits the following eight Noether symmetries and corresponding eight conserved quantities:

$$
\begin{aligned}
\Gamma_1 - \Gamma_5 \quad &\Rightarrow \quad I_1 = kq_1\dot{q}_2 - t\dot{q}_2 + q_2 - kq_2\dot{q}_1 \\
\Gamma_6 \quad &\Rightarrow \quad I_6 = m\dot{q}_2 \cos\left(\sqrt{\frac{k}{m}}t\right) + q_2\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}t\right) \\
\Gamma_7 \quad &\Rightarrow \quad I_7 = -q_2\sqrt{\frac{m}{k}} \cos\left(\sqrt{\frac{k}{m}}t\right) + m\dot{q}_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \\
\Gamma_8 \quad &\Rightarrow \quad I_8 = -\sqrt{\frac{m}{k}} \left(k^3\left(q_1^2 + q_2^2\right) - m + k\left(2m\dot{q}_1 + t^2\right)\right. \\
&\qquad -k^2\left(m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + 2tq_1\right)\right)\left(1 - 2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right) \\
&\qquad -4km\left(k(q_1 + \dot{q}_1 t) - t - k^2(q_1\dot{q}_1 + q_2\dot{q}_2)\right) \sin\left(\sqrt{\frac{k}{m}}t\right) \cos\left(\sqrt{\frac{k}{m}}t\right) \\
\Gamma_9 \quad &\Rightarrow \quad I_9 = \sqrt{\frac{m}{k}} \left(k^3\left(q_1^2 + q_2^2\right) - m + k\left(2m\dot{q}_1 + t^2\right)\right. \\
&\qquad -k^2\left(m\left(\dot{q}_1^2 + \dot{q}_2^2\right) + 2tq_1\right)\right) \sin\left(\sqrt{\frac{k}{m}}t\right) \cos\left(\sqrt{\frac{k}{m}}t\right) \\
&\qquad +km\left(k(q_1 + \dot{q}_1 t) - t - k^2(q_1\dot{q}_1 + q_2\dot{q}_2)\right)\left(1 - 2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right) \\
\Gamma_{10} \quad &\Rightarrow \quad I_{10} = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}k(q_1^2 + q_2^2) - tq_1 - \frac{m}{k}\dot{q}_1 + \frac{t^2}{2k} \\
\Gamma_{14} \quad &\Rightarrow \quad I_{14} = \sqrt{\frac{m}{k}}(kq_1 - t) \sin\left(\sqrt{\frac{k}{m}}t\right) + (k\dot{q}_1 - 1) \cos\left(\sqrt{\frac{k}{m}}t\right) m \\
\Gamma_{15} \quad &\Rightarrow \quad I_{15} = \sqrt{\frac{m}{k}}(kq_1 - t) \cos\left(\sqrt{\frac{k}{m}}t\right) - (k\dot{q}_1 - 1) \sin\left(\sqrt{\frac{k}{m}}t\right) m. \qquad (9)
\end{aligned}
$$

None of these Noether symmetries admits $f =$ constant. For example:

$$
\begin{aligned}
\Gamma_6 \quad &\Rightarrow \quad f = -q_2\sqrt{mk} \sin\left(\sqrt{\frac{k}{m}}t\right), \\
\Gamma_8 \quad &\Rightarrow \quad f = 2tq_1 - \frac{t^2}{2k} + \frac{m}{k^2} - k(q_1^2 + q_2^2) \\
&\qquad -2\sqrt{\frac{m}{k}}(kq_1 - t) \cos\left(\sqrt{\frac{k}{m}}t\right) \sin\left(\sqrt{\frac{k}{m}}t\right) \\
&\qquad -\left(4tq_1 - \frac{t^2}{k} + \frac{m}{k^2} - 2k(q_1^2 + q_2^2)\right) \sin^2\left(\sqrt{\frac{k}{m}}t\right). \qquad (10)
\end{aligned}
$$

Further details on system (7) can be found in [14]. Actually, we did not need eight first integrals; four would have been enough. A waste of effort? Not really, as we show next.

Quantizing with Noether symmetries

In [15] we proposed a quantization scheme that preserves the Noether symmetries of the underlying Lagrangian in order to construct the Schrödinger equation. If a system of second-order equations is considered, *i.e.*,

$$\ddot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{x} \in \mathrm{I\!R}^N, \tag{11}$$

that comes from a variational principle with a Lagrangian of first order, then our method can be summarized as follows:

1. Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} W_k(t, \mathbf{x})\partial_{x_k}.$$

2. Among them find the Noether symmetries

$$\Gamma = V(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} V_k(t, \mathbf{x})\partial_{x_k}.$$

   This may require searching for the Lagrangian yielding the maximum possible number of Noether symmetries [22, 23, 27, 28].

3. Construct the Schrödinger equation[10] admitting these Noether symmetries as Lie symmetries, namely

$$2i\Psi_t + \sum_{k,j=1}^{N} f_{kj}(\mathbf{x})\Psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\mathbf{x})\Psi_{x_k} + f_0(\mathbf{x})\Psi = 0 \tag{12}$$

   with Lie symmetries

$$\Omega = V(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} V_k(t, \mathbf{x})\partial_{x_k} + G(t, \mathbf{x}, \Psi)\partial_\Psi$$

   without adding any other symmetries apart from the two symmetries that are present in any linear homogeneous partial differential equation[11], namely

$$\Psi\partial_\Psi, \qquad \alpha(t, \mathbf{x})\partial_\Psi,$$

   where $\alpha = \alpha(t, \mathbf{x})$ is any solution of the Schrödinger equation (12).

[10] We assume $\hbar = 1$ without loss of generality.

[11] In the following we will refer to those two symmetries as the homogeneity and linearity symmetries.

For example, let us consider the well-known problem of a charged particle in a uniform magnetic field in the plane. The corresponding classical Lagrangian is

$$L = \frac{1}{2}\left((\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y})\right), \tag{13}$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}. \tag{14}$$

The Schrödinger equation was determined by Sir Charles Galton Darwin[12] in [2] to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \tag{15}$$

The Lie-symmetry algebra admitted by the linear system (14) has dimension fifteen, and the classical Lagrangian (13) admits eight Noether symmetries generated by the following operators:

$$\begin{aligned}
X_1 &= \cos(\omega t)\partial_t - \frac{1}{2}\left(\sin(\omega t)\omega x + \cos(\omega t)\omega y\right)\partial_x + \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_y,\\
X_2 &= -\sin(\omega t)\partial_t - \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_x - \frac{1}{2}(\sin(\omega t)\omega x + \cos(\omega t)\omega y)\partial_y,\\
X_3 &= \partial_t,\\
X_4 &= -y\partial_x + x\partial_y,\\
X_5 &= -\sin(\omega t)\partial_x + \cos(\omega t)\partial_y,\\
X_6 &= -\cos(\omega t)\partial_x - \sin(\omega t)\partial_y,\\
X_7 &= \partial_y,\\
X_8 &= \partial_x.
\end{aligned} \tag{16}$$

The Schrödinger equation (15) admits an infinite Lie-symmetry algebra[13] generated by the operator $\alpha(t, x, y)\partial_\psi$, where α is any solution of the equation itself, and also a finite-dimensional

[12] A grandson of Charles Robert Darwin.
[13] This is true for any linear partial differential equations.

Lie-symmetry algebra generated by the following operators:

$$\begin{aligned} Y_1 &= X_1 + \frac{1}{4}\left(2\sin(\omega t)\omega - i\cos(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi, \\ Y_2 &= X_2 + \frac{1}{4}\left(2\cos(\omega t)\omega + i\sin(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi, \\ Y_3 &= X_3, \\ Y_4 &= X_4, \\ Y_5 &= X_5 - \frac{1}{2}\omega\left(x\cos(\omega t) + y\sin(\omega t)\right)\partial_\psi, \\ Y_6 &= X_6 + \frac{1}{2}\omega\left(x\sin(\omega t) - y\cos(\omega t)\right)\partial_\psi, \\ Y_7 &= X_7 + \frac{i}{2}\omega x\partial_\psi, \\ Y_8 &= X_8 - \frac{i}{2}\omega y\partial_\psi, \\ Y_9 &= \psi\partial_\psi. \end{aligned} \tag{17}$$

This known example supported our method, namely that the Schrödinger equation admits a finite Lie-symmetry algebra that corresponds to the Noether symmetries admitted by the classical Lagrangian plus the symmetry Y_9 admitted by any homogeneous linear partial differential equation.

We successfully applied our method to various classical problems: alternative Hamiltonian of an harmonic oscillator [15], second-order Riccati equation [16], dynamics of a charged particle in a uniform magnetic field and a non-isochronous Calogero goldfish system [17], an equation related to a Calogero goldfish equation [18], two nonlinear equations somewhat related to the Riemann problem [19], a Liénard I nonlinear oscillator [8], a family of Liénard II nonlinear oscillators [9], N planar rotors and an isochronous Calogero goldfish system [20], the motion of a free particle and that of an harmonic oscillator on a double cone [10].

Here, we would like to recall what we wrote in [17]. Let us consider the equations of motions of two uncoupled harmonic oscillators, *i.e.*,

$$\ddot{q}_1 = -\omega^2 q_1, \qquad \ddot{q}_2 = -\omega^2 q_2. \tag{18}$$

It is known [33] that this system possesses two Lagrangians (actually, an infinite number [3]): one is the usual well-known physical Lagrangian, *i.e.*,

$$L_1 = \tfrac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right) \tag{19}$$

that admits the following eight Noether symmetries:

$$\begin{aligned} Y_1 = -q_2\partial_{q_1} + q_1\partial_{q_2},\ Y_2 = \cos(2\omega t)\partial_t - \sin(2\omega t)\omega q_1\partial_{q_1} - \sin(2\omega t)\omega q_2\partial_{q_2}, \\ Y_3 = -\sin(2\omega t)\partial_t - \cos(2\omega t)\omega q_1\partial_{q_1} - \cos(2\omega t)\omega q_2\partial_{q_2},\ Y_4 = \partial_t,\ Y_5 = \cos(\omega t)\partial_{q_2}, \\ Y_6 = -\sin(\omega t)\partial_{q_2},\ Y_7 = \cos(\omega t)\partial_{q_1},\ Y_8 = -\sin(\omega t)\partial_{q_1}, \end{aligned} \tag{20}$$

and another nonphysical Lagrangian that can be found in [33] p.122:

$$L_2 = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2 \tag{21}$$

that admits the following eight Noether symmetries:

$$\tilde{Y}_1 = -q_1\partial_{q_1} + q_2\partial_{q_2}, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5, \quad Y_6, \quad Y_7, \quad Y_8. \tag{22}$$

Both Lagrangians admit the maximal number of Noether symmetries, albeit they differ just by one, namely Y_1 instead of $\tilde{Y}_1$. Indeed the rotational symmetry Y_1 is an essential physical property of two uncoupled harmonic oscillators since it yields the conservation of angular momentum. From this example we can infer the obvious conjecture that the physical Lagrangian admits the maximum number of Noether symmetries that also lead to the essential physical conservation laws[14].

Indeed if we apply the method described in this paper, i.e using the Noether symmetries (20) admitted by the physical Lagrangian L_1, then the known Schrödinger equation for the two-dimensional oscillator is obtained, *i.e.*,

$$2i\psi_t + \psi_{q_1q_1} + \psi_{q_2q_2} - \omega^2(q_1^2 + q_2^2)\psi = 0. \tag{23}$$

Instead no linear partial differential equation in the dependent variable $\psi(t, q_1, q_2)$ of the three independent variables t, q_1, q_2, *i.e.*,

$$f_{11}\psi_{tt} + f_{12}\psi_{tq_1} + f_{13}\psi_{tq_2} + f_{22}\psi_{q_1q_1} + f_{23}\psi_{q_1q_2} + f_{33}\psi_{q_2q_2} + f_1\psi_t + f_2\psi_{q_1} + f_3\psi_{q_2} + f_0\psi = 0 \tag{24}$$

where $f_{ij}(i, j = 1, 2, 3)$, $f_k(k = 0, 1, 2, 3)$ are arbitrary functions of t, q_1, q_2, admits as Lie finite symmetries the eight Noether symmetries (22) of the nonphysical Lagrangian L_2.

Thus the physical Lagrangian is indeed the one that directly yields to quantization without any further trick.

Noether ghostbuster

Some simple linear equations of classical mechanics yield serious problems when quantization à la Dirac is undertaken since states with negative norm, commonly called 'ghosts', appear. These ghosts are not 'Faddeev-Popov ghosts'. In fact as Faddeev wrote [4]: *In the terminology of theoretical physics, the term 'ghost' is used to identify an object that has no real physical meaning. The name 'Faddeev-Popov ghosts' is given to the fictitious fields that were originally introduced in the construction of a manifestly Lorentz covariant quantization of the YangMills field. . . . One needs a suitable mechanism in order to get rid of the nonphysical degrees of freedom. Introducing fictitious fields, the ghosts, is one way of achieving this goal.* Therefore the 'Faddeev-Popov ghosts' are 'nice ghosts' and they have nothing to do with the 'bad ghosts' that we are dealing with.

[14] We remark that these conservation laws are obviously too many – and therefore some are functionally dependent on the others – for the integration of the classical system, but they are just right in number and physical quality for the corresponding Schrödinger equation.

A 'ghostbuster' based on the preservation of the Noether's symmetries of the original Lagrangian was proposed in [25], [24] in order to achieve a quantization without ghosts. The method was applied to the fourth-order field-theoretic model of Pais-Uhlenbeck [32]:

$$\ddddot{z} + \left(\Omega_1^2 + \Omega_2^2\right) \ddot{z} + \Omega_1^2\Omega_2^2 z = 0, \tag{25}$$

namely the Euler-Lagrange equation corresponding to the second-order Lagrangian

$$L = \frac{1}{2}\gamma \left\{\ddot{z}^2 - \left(\Omega_1^2 + \Omega_2^2\right) \dot{z}^2 + \Omega_1^2\Omega_2^2 z^2\right\}. \tag{26}$$

In [15] we reinforced the usefulness of the method by applying it to a third-order Lagrangian that was presented in [11], *i.e.*,

$$L = \frac{1}{2M^4}\Big(x'''^2 - (\omega^2 + 2\cos(2\Theta)M^2)x''^2 - M^2(M^2 + 2\cos(2\Theta)\omega^2)x'^2 - M^4\omega^2x^2\Big). \tag{27}$$

This Lagrangian leads to the following sixth-order Euler-Lagrange equation:

$$\frac{1}{M^4}x^{(vi)} + \frac{2}{M^2}\left(\cos(2\Theta) + \frac{\omega^2}{2M^2}\right)x^{(iv)} - \left(1 + \frac{2\omega^2}{M^2}\cos(2\Theta)\right)x'' + \omega^2 x = 0. \tag{28}$$

In [11] the Ostrogradsky's method was used, as usual, in order to write the Hamiltonian corresponding to the Lagrangian (27), and then the quantization of that Hamiltonian yielded both zero-norm state vectors with complex energy and others with negative norm. Instead, we have shown that the method presented in [25], [24] yields the correct Hamiltonian, namely that that preserves the Noether symmetries of the original Lagrangian, and quantization without ghosts is achieved. Let's then consider the sixth-order equation (28) and corresponding Lagrangian (27).

We summarize Ostrogradsky's method for recasting a problem from the calculus of variations in one independent variable with a higher-order Lagrangian into Hamiltonian form [31], [34].

Let $L\left(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}\right)$, where x can be a multivariable and n a multi-index in a general treatment, but we confine our attention to a single dependent variable, which may occur up to the nth derivative with respect to the independent variable, t, be a Lagrangian for which the Euler-Lagrange equation of the calculus of variations is

$$0 = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \ldots + (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\partial L}{\partial x^{(n)}}\right). \tag{29}$$

To obtain a Hamiltonian representation a first-order Lagrangian is required. Ostrogradsky defines the momenta as

$$\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{x}}\right) + \ldots + (-1)^{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left(\frac{\partial L}{\partial x^{(n)}}\right)\\
p_2 &= \frac{\partial L}{\partial \ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dddot{x}}\right) + \ldots + (-1)^{n-2}\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}\left(\frac{\partial L}{\partial x^{(n)}}\right)\\
&\vdots\\
p_n &= \frac{\partial L}{\partial x^{(n)}},
\end{aligned} \tag{30}$$

which ensures that all derivatives present in the Lagrangian are covered. The canonical coordinates are defined according to

$$q_1 = x,\ q_2 = \dot{x},\ \ldots,\ q_n = x^{(n-1)}. \tag{31}$$

The Hamiltonian function is then defined according to

$$H = -L + p_1 q_2 + p_2 q_3 + \ldots + p_{n-1} q_n + p_n x^{(n)}, \tag{32}$$

which reduces to the standard prescription when the Lagrangian is of the first order. When Hamilton's principle is applied, we obtain the standard equations for a Hamiltonian describing a system of n degrees of freedom, namely

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}. \tag{33}$$

The formalism is fitting and yet the transition to quantum mechanics has led to a flawed description of the physical properties of the original system as in [32], and [11].

In order to simplify the expressions we introduce three new parameters $\omega_1, \omega_2, \omega_3$ such that equation (28) can be rewritten as

$$x^{(vi)} = -(\omega_1^2 + \omega_2^2 + \omega_3^2)x^{(iv)} - (\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2)x'' - \omega_1^2\omega_2^2\omega_3^2 x \tag{34}$$

and its corresponding Lagrangian (27) as

$$L = \frac{1}{2}\Big(x'''^2 - (\omega_1^2 + \omega_2^2 + \omega_3^2)x''^2 + (\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2)x'^2 - \omega_1^2\omega_2^2\omega_3^2 x^2\Big) + \frac{\mathrm{d}g}{\mathrm{d}t}, \tag{35}$$

with $g = g(t, x, x', x'')$ the gauge function.

Equation (34) admits an eight-dimensional Lie point-symmetry algebra generated by the following operators:

$$\Gamma_1 = \sin(\omega_1 t)\partial_x, \quad \Gamma_2 = \cos(\omega_1 t)\partial_x, \quad \Gamma_3 = \sin(\omega_3 t)\partial_x, \quad \Gamma_4 = \cos(\omega_3 t)\partial_x,$$
$$\Gamma_5 = \sin(\omega_2 t)\partial_x, \quad \Gamma_6 = \cos(\omega_2 t)\partial_x, \quad \Gamma_7 = x\partial_x, \quad \Gamma_8 = \partial_t,$$

while the Lagrangian (35) admits seven Noether point symmetries and consequently seven first integrals, namely:

$$
\begin{aligned}
\Gamma_1 \;\; &\Rightarrow \;\; Int_1 = \Big(\omega_2^2\omega_3^2 x + (\omega_2^2+\omega_3^2)x'' + x^{(iv)}\Big)\cos(\omega_1 t)\omega_1 \\
&\qquad -\Big(\omega_2^2\omega_3^2 x' + (\omega_2^2+\omega_3^2)x''' + x^{(v)}\Big)\sin(\omega_1 t) \qquad (36)\\
\Gamma_2 \;\; &\Rightarrow \;\; Int_2 = \Big(\omega_2^2\omega_3^2 x + (\omega_2^2+\omega_3^2)x'' + x^{(iv)}\Big)\sin(\omega_1 t)\omega_1 \\
&\qquad +\Big(\omega_2^2\omega_3^2 x' + (\omega_2^2+\omega_3^2)x''' + x^{(v)}\Big)\cos(\omega_1 t) \qquad (37)\\
\Gamma_3 \;\; &\Rightarrow \;\; Int_3 = \Big(\omega_1^2\omega_2^2 x + (\omega_1^2+\omega_2^2)x'' + x^{(iv)}\Big)\cos(\omega_3 t)\omega_3 \\
&\qquad -\Big(\omega_1^2\omega_2^2 x' + (\omega_1^2+\omega_2^2)x''' + x^{(v)}\Big)\sin(\omega_3 t) \qquad (38)\\
\Gamma_4 \;\; &\Rightarrow \;\; Int_4 = \Big(\omega_1^2\omega_2^2 x + (\omega_1^2+\omega_2^2)x'' + x^{(iv)}\Big)\sin(\omega_3 t)\omega_3 \\
&\qquad +\Big(\omega_1^2\omega_2^2 x' + (\omega_1^2+\omega_2^2)x''' + x^{(v)}\Big)\cos(\omega_3 t) \qquad (39)\\
\Gamma_5 \;\; &\Rightarrow \;\; Int_5 = \Big(\omega_1^2\omega_3^2 x + (\omega_1^2+\omega_3^2)x'' + x^{(iv)}\Big)\cos(\omega_2 t)\omega_2 \\
&\qquad -\Big(\omega_1^2\omega_3^2 x' + (\omega_1^2+\omega_3^2)x''' + x^{(v)}\Big)\sin(\omega_2 t) \qquad (40)\\
\Gamma_6 \;\; &\Rightarrow \;\; Int_6 = \Big(\omega_1^2\omega_3^2 x + (\omega_1^2+\omega_3^2)x'' + x^{(iv)}\Big)\sin(\omega_2 t)\omega_2 \\
&\qquad +\Big(\omega_1^2\omega_3^2 x' + (\omega_1^2+\omega_3^2)x''' + x^{(v)}\Big)\cos(\omega_2 t) \qquad (41)\\
\Gamma_8 \;\; &\Rightarrow \;\; Int_8 = \tfrac{1}{2}\Big(-x'''^2 + (\omega_1^2+\omega_2^2+\omega_3^2)x'x''' - (\omega_1^2+\omega_2^2+\omega_3^2)x''^2 \\
&\qquad +(\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2)x'^2 + (\omega_1^2\omega_2^2\omega_3^2)x^2\Big). \qquad (42)
\end{aligned}
$$

En passant we remark that these first integrals, except Int_8, could not be obtained if the gauge function g were taken to be a constant.

If we suitably combine the first integrals I_1, I_2, I_3, I_4, I_5 and I_6 as

$$\tfrac{1}{2}(Int_1^2 + Int_2^2 + Int_3^2 + Int_4^2 + Int_5^2 + Int_6^2),$$

then the autonomous first integral

$$
\begin{aligned}
I_0 = \frac{1}{2}\Big[&\big(\omega_2^2\omega_3^2 x + (\omega_2^2+\omega_3^2)x'' + x^{(iv)}\big)^2\omega_1^2 + \big(\omega_2^2\omega_3^2 x' + (\omega_2^2+\omega_3^2)x''' + x^{(v)}\big)^2 \\
&+ \big(\omega_1^2\omega_2^2 x + (\omega_1^2+\omega_2^2)x'' + x^{(iv)}\big)^2\omega_3^2 + \big(\omega_1^2\omega_2^2 x' + \left(\omega_1^2+\omega_2^2\right)x''' + x^{(v)}\big)^2 \\
&+ \Big(\omega_1^2\omega_3^2 x + (\omega_1^2+\omega_3^2)x'' + x^{(iv)}\big)^2\omega_2^2 + \big(\omega_1^2\omega_3^2 x' + (\omega_1^2+\omega_3^2)x''' + x^{(v)}\big)^2\Big] \qquad (43)
\end{aligned}
$$

is obtained. If we make the obvious transformations

$$
\begin{aligned}
q_1 &= \omega_2^2\omega_3^2 x + (\omega_2^2+\omega_3^2)x'' + x^{(iv)}\\
q_2 &= \omega_1^2\omega_3^2 x + (\omega_1^2+\omega_3^2)x'' + x^{(iv)}\\
q_3 &= \omega_1^2\omega_2^2 x + (\omega_1^2+\omega_2^2)x'' + x^{(iv)}\\
p_1 &= \omega_2^2\omega_3^2 x' + (\omega_2^2+\omega_3^2)x''' + x^{(v)}\\
p_2 &= \omega_1^2\omega_3^2 x' + (\omega_1^2+\omega_3^2)x''' + x^{(v)}\\
p_3 &= \omega_1^2\omega_2^2 x' + (\omega_1^2+\omega_2^2)x''' + x^{(v)},
\end{aligned}
$$

from (43) we obtain the Hamiltonian

$$H \equiv I_0 = \frac{1}{2}(p_1^2+p_2^2+p_3^2+\omega_1^2q_1^2+\omega_2^2q_2^2+\omega_3^2q_3^2), \tag{44}$$

and the corresponding canonical equations are

$$
\begin{aligned}
\dot q_1 = p_1, \quad \dot q_2 = p_2, \quad \dot q_3 = p_3,\\
\dot p_1 = -\omega_1^2 q_1, \quad \dot p_2 = -\omega_2^2 q_2, \quad \dot p_3 = -\omega_3^2 q_3.
\end{aligned} \tag{45}
$$

This is the right Hamiltonian for the quantization of the sixth-order equation (28).

The application of the Legendre transformation to the Hamiltonian (44) gives the Lagrangian

$$L = \tfrac{1}{2}[\dot q_1^2+\dot q_2^2+\dot q_3^2 - (\omega_1^2q_1^2+\omega_2^2q_2^2+\omega_3^2q_3^2)] \tag{46}$$

and corresponding Lagrangian equations

$$
\begin{aligned}
\ddot q_1 &= -\omega_1^2 q_1,\\
\ddot q_2 &= -\omega_2^2 q_2,\\
\ddot q_3 &= -\omega_3^2 q_3,
\end{aligned} \tag{47}
$$

which admit a ten-dimensional Lie point symmetry algebra[15] generated by the following operators

$$
\begin{aligned}
\Lambda_1 = \partial_t, \quad \Lambda_2 = q_1\partial_{q_1}, \quad \Lambda_3 = \cos(\omega_1 t)\partial_{q_1}, \quad \Lambda_4 = -\sin(\omega_1 t)\partial_{q_1},\\
\Lambda_5 = q_2\partial_{q_2}, \quad \Lambda_6 = \cos(\omega_2 t)\partial_{q_2}, \quad \Lambda_7 = -\sin(\omega_2 t)\partial_{q_2},\\
\Lambda_8 = q_3\partial_{q_3}, \quad \Lambda_9 = \cos(\omega_3 t)\partial_{q_3}, \quad \Lambda_{10} = -\sin(\omega_3 t)\partial_{q_3}.
\end{aligned} \tag{48}
$$

We note that the Lagrangian (46) possesses *seven* Noether point symmetries and consequently

[15]System (47) admits a twenty-four-dimensional Lie point-symmetry algebra if $\omega_1=\omega_2=\omega_3$.

seven first integrals, namely

$$\begin{aligned}
\Lambda_1 &\implies I_1 = \tfrac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \omega_3^2 q_3^2 + \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)\\
\Lambda_3 &\implies I_3 = \sin(\omega_1 t)\omega_1 q_1 + \cos(\omega_1 t)\dot{q}_1\\
\Lambda_4 &\implies I_4 = \cos(\omega_1 t)\omega_1 q_1 - \sin(\omega_1 t)\dot{q}_1\\
\Lambda_6 &\implies I_6 = \sin(\omega_2 t)\omega_2 q_2 + \cos(\omega_2 t)\dot{q}_2\\
\Lambda_7 &\implies I_7 = \cos(\omega_2 t)\omega_2 q_2 - \sin(\omega_2 t)\dot{q}_2\\
\Lambda_9 &\implies I_9 = \sin(\omega_3 t)\omega_3 q_3 + \cos(\omega_3 t)\dot{q}_3\\
\Lambda_{10} &\implies I_{10} = \cos(\omega_3 t)\omega_3 q_3 - \sin(\omega_3 t)\dot{q}_3.
\end{aligned} \tag{49}$$

This is in agreement with the Noether symmetries and first integrals (42) admitted by the Lagrangian (35).

The method that we have used eliminates the bad ghosts and consists of the following steps:

- Find the Lie symmetries of the higher-order Lagrangian equation.
- Find the Noether symmetries and the corresponding first integrals.
- Construct the Hamiltonian from the first integrals by combining all of them except the autonomous one.
- Apply the Legendre transform and verify that you get the same number of Noether symmetries.

Conclusion: the true ghost-busters are Lie and Noether.

Lagrangian and Noether symmetries for systems without Lagrangians

In [26] we examined the properties of two of the examples given by Douglas [3] of a system which does not possess a Lagrangian. The first two-dimensional system is a special instance of the class Case I in Douglas and has been mentioned by Gitman and Kupriyanov [7],

$$\left\{\begin{array}{rcl} \ddot{x}+\dot{y} &=& 0,\\ \ddot{y}+y &=& 0. \end{array}\right. \tag{50}$$

Actually, a quite similar system, namely

$$\left\{\begin{array}{rcl} \ddot{x}-\dot{y} &=& 0,\\ \\ \ddot{y}-y &=& 0, \end{array}\right.$$

was proposed in 1923 by E.T. Whittaker as stated by Bateman in his famous paper [1]. This system does not possess a Lagrangian and defeats Bateman's method of adding a mirror system to it. The second system is

$$\left\{\begin{array}{rcl} \ddot{x} &=& x^2+y^2,\\ \ddot{y} &=& 0, \end{array}\right. \tag{51}$$

and is an example the class Case IIIB in Douglas. In both cases we demonstrated the existence of several Lagrangians by means of the simple expedient of a reformulation of the systems.

We can construct a Lagrangian in three different ways.

(1) Raise the order to one equation.

The system (50) can be written as a single fourth-order equation by the stratagem of differentiating the second with respect to t and then substituting for x from the first to obtain

$$\ddddot{x} + \ddot{x} = 0. \tag{52}$$

This equation satisfies the conditions of Fels [6] and its unique second-order Lagrangian is the obvious one [21], namely

$$L = \tfrac{1}{2}\left(\ddot{x}^2 - \dot{x}^2\right). \tag{53}$$

We applied Noether's Theorem to this Lagrangian and obtained the following five Noether symmetries and associated first integrals:

$$\begin{aligned}
\Gamma_1 &= \partial_t & \Rightarrow\quad & I_1 = \ddot{x}^2 - 2\dot{x}\,\dddot{x} - \dot{x}^2\\
\Gamma_2 &= \partial_x & \Rightarrow\quad & I_2 = \dddot{x} + \dot{x}\\
\Gamma_3 &= t\partial_x & \Rightarrow\quad & I_3 = x - \ddot{x} + t\left(\dot{x} + \dddot{x}\right)\\
\Gamma_4 &= \sin t\partial_x & \Rightarrow\quad & I_4 = \ddot{x}\sin t + \dddot{x}\cos t\\
\Gamma_5 &= \cos t\partial_x & \Rightarrow\quad & I_5 = \ddot{x}\cos t - \dddot{x}\sin t.
\end{aligned}$$

(2) Write an equivalent system of first-order equations, lower the order, and then raise it once.

We rewrite system (50) as the set of four first-order equations

$$\dot{w}_1 = w_2 \tag{54}$$
$$\dot{w}_2 = -w_4 \tag{55}$$
$$\dot{w}_3 = w_4 \tag{56}$$
$$\dot{w}_4 = -w_3, \tag{57}$$

where we have written $x = w_1$ and $y = w_3$.

System (54)-(57) is autonomous and we can reduce its order by one with the choice of one of the dependent variables in (54)-(57) as a new independent variable. We choose w_1 and revert to the original symbol x. Then system (54)-(57) becomes

$$\frac{\mathrm{d}w_2}{\mathrm{d}x} = -\frac{w_4}{w_2} \tag{58}$$
$$\frac{\mathrm{d}w_3}{\mathrm{d}x} = \frac{w_4}{w_2} \tag{59}$$
$$\frac{\mathrm{d}w_4}{\mathrm{d}x} = -\frac{w_3}{w_2}. \tag{60}$$

From (58)-(59) it is obvious that $w_3 + w_2 = r_3$ is a constant, $r_3 = r_0$. Therefore $w_3 = r_0 - w_2$. We use (58) to eliminate w_4 through

$$w_4 = -\frac{\mathrm{d}w_2}{\mathrm{d}x} w_2.$$

Then (60) becomes

$$\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} = -\frac{1}{w_2}\left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2 - \frac{1}{w_2} + \frac{r_0}{w_2^2} \tag{61}$$

which is a single second-order equation for which a Lagrangian exists[16]. It is[17] [34]

$$L = \tfrac{1}{2}\, w_2^2 \left(\frac{\mathrm{d}w_2}{\mathrm{d}x}\right)^2 - \tfrac{1}{2} w_2^2 + r_0 w_2, \tag{62}$$

which admits one obvious Noether symmetry and first integral.

(3) Write an equivalent system of first-order equations, and then raise the order twice. Moreover, we can transform system (54)-(57) into a system of two second-order equations which admits a Lagrangian. We use (54) to eliminate w_2 through

$$w_2 = \dot{w}_1,$$

and (57) to eliminate w_3 through

$$w_3 = -\dot{w}_4.$$

Then (55) and (56) yield

$$\begin{aligned} \ddot{w}_1 &= -w_4, \\ \ddot{w}_4 &= -w_4, \end{aligned} \tag{63}$$

which is a two-dimensional system for which an infinite number of Lagrangian exists [3]. The method we described in [23] yields the following Lagrangian for system (63):

$$L = \tfrac{1}{2}\left(\dot{w}_1^2 - 2\dot{w}_1\dot{w}_4 + 2\dot{w}_4^2 - w_4^2\right), \tag{64}$$

which admits five Noether symmetries and corresponding first integrals:

$$\begin{array}{lll} \Gamma_1 = \partial_t & \Rightarrow & I_1 = \tfrac{1}{2}\left(\dot{w}_1^2 - 2\dot{w}_1\dot{w}_4 + 2\dot{w}_4^2 + w_4^2\right) \\ \Gamma_2 = \sin(t)\left(\partial_{w_1} + \partial_{w_4}\right) & \Rightarrow & I_2 = \cos(t) w_4 - \sin(t)\dot{w}_4 \\ \Gamma_3 = \cos(t)\left(\partial_{w_1} + \partial_{w_4}\right) & \Rightarrow & I_3 = -\cos(t)\dot{w}_4 - \sin(t) w_4 \\ \Gamma_4 = \partial_{w_1} & \Rightarrow & I_4 = \dot{w}_4 - \dot{w}_1 \\ \Gamma_5 = t\partial_{w_1} & \Rightarrow & I_5 = w_1 - w_4 - t(\dot{w}_1 - \dot{w}_4). \end{array}$$

[16] In terms of the original variables $w_2 = \dot{x}$.
[17] A Jacobi Last Multiplier of (61) is w_2^2.

Final remarks

We have exemplified different instances where Noether's first theorem plays a fundamental role: finding conservation laws, going from classical to quantum mechanics, eliminating 'ghosts'. Also, we have shown that Noether symmetries are ubiquitous, namely they exist even for systems without a Lagrangian.

Of course, our examples are not exhaustive of the many fields where Noether's theorems can be used. In the last one hundred years, many mathematicians, physicists, and mathematical physicists have applied Noether's theorems in their research. Also, at the beginning of the next hundred years, many mathematicians, physicists, and mathematical physicists are applying Noether's theorems in their research. We are all saying: thank you, Emmy Noether.

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Noether's Cosmology

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Noether's symmetries have a prominent role in any branch of physics. Here we discuss their role in cosmology. In particular, we consider minisuperspace cosmological models, showing that the existence of symmetries, and then conserved quantities, not only allow one to obtain exact cosmological solutions but also give selection rules by which it is possible to recover observable behaviors in cosmic evolution. In this sense, symmetries are related to physically reliable models. Some specific models are worked out starting from given theories of gravity.

1 Introduction

Centenary celebrations of Noether's theorem are an opportunity to underline the work and the relevance of this great woman who marked the history of mathematics as well as of science. It is important to stress that the concept of symmetry has evolved and improved along the last century: the transition from global continuous symmetries (such as translation and rotation) to local continuous symmetries, and, in particular, local gauge symmetries, the addition of the internal symmetries to the space-time symmetries, the connection between charge conjugation and time reversal with the discrete symmetry of parity, symmetries in quantum field theory point out that the concept of symmetry is unavoidable in mathematics, physics and other sciences [1]. The leitmotiv of this interest is that symmetries have become a key tool for the construction of any theory at fundamental level. For example, it is worth noting the role of the principle of relativity in special relativity and the role of covariance as a symmetry principle in general relativity as well as the role of groups in quantum field theory or in particle physics. In particular, the symmetry principle plays a fundamental role in the construction of the standard model of particles.

In this context, the work by Emmy Noether is of central importance since she provided a general proof that, from the continuous symmetries of a given theory, we can derive the conservation laws of that theory. In particular she discovered the relation between symmetries of dynamical systems and their first integrals, *i.e.*, physical quantities which remain constant during the evolution of the system and are related to fundamental physical quantities such as energy, linear momentum, angular momentum, and so on [2, 3].

Recently, Noether symmetries assumed a prominent role also in cosmology because they allow one to select cosmological models whose parameters are observable quantities [4]. This statement means that Noether symmetries are capable of selecting physical models in a given class of theories. This issue is crucial in so-called quantum cosmology, that branch of cosmology aimed to select initial conditions from which classical (and then observable) universes can emerge. Specifically, according to the Hartle criterion [5], the correlation among physical variables selects classical trajectories which are solutions of the cosmological field equations. This is realized if the solution of the Wheeler-DeWitt equation shows oscillating behavior. Such a solution is called the wave function of the universe. It is possible to show that the Hartle criterion is related to Noether symmetries. In other words, it can be shown that if symmetries exist the wave function of the universe presents oscillating behaviors [6] and then the presence of Noether symmetries allows the emergence of classically observable universes.

Here, without pretending to be complete, we want to briefly sketch the use of Noether's theorem in cosmology. In Sec. 2, we discuss the so-called minisuperspace approach to quantum cosmology. Noether symmetries for Lagrangian and Hamiltonian dynamical systems, related to cosmological models, are discussed in Sec. 3. Applications to specific cosmological models, derived from some theories gravity, are presented in Sec. 4. In Sec. 5, we discuss the results and draw conclusions.

2 Minisuperspaces in Cosmology

In order to discuss Noether symmetries in cosmology, we have to take into account the concept of *minisuperspace* which is the finite dimensional configuration space of dynamical variables describing a cosmological model. In general, a minisuperspace is a restriction of *superspace*, the infinite-dimensional configuration space of a given theory of gravity, where the forms of the metric and matter fields are fixed in a given family (e.g. homogeneous, spherical, cylindrical, etc.). The simplest models of minisuperspaces are constructed according to the cosmological principle by choosing homogeneous and isotropic metrics and matter fields. Other forms of minisuperspaces can be achieved following the Bianchi classification of universes [7].

A simple minisuperspace can be provided assuming a homogeneous lapse function $N = N(t)$, the shift functions $N^i = 0$ set to zero and a 4-metric of the form

$$ds^2 = -N^2(t)dt^2 + h_{ij}(\mathbf{x}, t)dx^i dx^j \,, \tag{65}$$

where h_{ij} is a 3-metric that is homogeneous and described by a finite number of functions of t, $q^\alpha(t)$, where $\alpha = 0, 1, 2 \cdots (n-1)$(see [6] and references therein). We can recast the Hilbert-Einstein action, where the Lagrangian is $\mathcal{L} = \sqrt{-g}\,[R - 2\Lambda]$ with R the Ricci curvature scalar and Λ the cosmological constant, as

$$\mathcal{S}[h_{ij}, N, N^i] \;=\; \frac{m_P^2}{16\pi}\int dt\, d^3x\, N\sqrt{h}\left[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\right] \,, \tag{66}$$

where K^{ij} is the extrinsic curvature, K its trace, ${}^{(3)}R$ the curvature and $\sqrt{h}$ the determinant of the spatial 3-manifold, respectively. In general, one gets[18]

$$\mathcal{S}[q^\alpha(t), N(t)] \;=\; \int_0^1 dtN\left[\frac{1}{2N^2}f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)\right] \equiv \int \mathcal{L}dt \,, \tag{67}$$

assembling together kinetic and configuration terms (see [6] for details). The last equation has the form of a relativistic point-particle action where the particles move on an n-dimensional

[18] Here, we have assumed as signature $(-,+,+,+)$. The function $f_{\alpha\beta}(q)$ is the reduced DeWitt metric, and the integration of t can range from 0 to 1 by shifting t and rescaling the lapse function.

curved space-time with a self-interaction potential. The variation with respect to q^α gives the equations of motion

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right)+\frac{1}{N^2}\Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma+f^{\alpha\beta}\frac{\partial U}{\partial q^\beta}=0\,, \tag{68}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols derived from the metric $f_{\alpha\beta}$. Varying with respect to N, one gets

$$\frac{1}{2N^2}f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta+U(q)=0\,, \tag{69}$$

which is a constraint equation.

Eqs(68) and (69) describe geodesic motion in minisuperspace with a forcing term. The general solution of (68), (69) requires $(2n-1)$ arbitrary parameters to be found. Eqs. (68) and (69) have to be equivalent, respectively, to the 00 and ij components of the Einstein field equations. This statement is not guaranteed since assuming a choice for the metric into the action and then taking variations to derive the minisuperspace field equations does not necessarily yield the same field equations. However, it holds in several cases, in particular for Bianchi models.

In order to find the Hamiltonian, the canonical momenta have to be defined, that is,

$$p_\alpha=\frac{\partial\mathcal{L}}{\partial\dot{q}^\alpha}=f_{\alpha\beta}\frac{\dot{q}^\beta}{N}\,, \tag{70}$$

and the canonical Hamiltonian is

$$\mathcal{H}_c=p_\alpha\dot{q}^\alpha-\mathcal{L}=N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta+U(q)\right]\equiv N\mathcal{H}\,, \tag{71}$$

where $f^{\alpha\beta}(q)$ is the inverse metric on minisuperspace. The action in Hamiltonian form is

$$\mathcal{S}=\int_0^1 dt\,[p_\alpha\dot{q}^\alpha-N\mathcal{H}]\ . \tag{72}$$

A Lagrange multiplier is the lapse function N and then the Hamiltonian constraint has to be

$$\mathcal{H}(q^\alpha,p_\alpha)=\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta+U(q)=0\,. \tag{73}$$

If we want to deal with quantum cosmology, the canonical quantization procedure requires a time-independent wave function $\Psi(q^\alpha)$ that has to be annihilated by the quantum operator corresponding to the classical constraint (73). In other words, we need cosmological models that have to be quantized in view of discussing the initial condition from which our observable universe emerged.

The procedure gives rises to the Wheeler-DeWitt equation,

$$\hat{\mathcal{H}}(q^\alpha,-i\frac{\partial}{\partial q^\alpha})\Psi(q^\alpha)=0\,, \tag{74}$$

where $\Psi(q^\alpha)$ is the so-called *wave function of the universe*, that is, the function indicating if cosmological observables are correlated or not [19]. Since the metric $f^{\alpha\beta}$ depends on q there is a factor ordering issue in (74). This may be solved by requiring that the quantization procedure is covariant in minisuperspace, that is, unchanged by field redefinitions of the 3-metric and matter fields, $q^\alpha \to \tilde{q}^\alpha(q^\alpha)$. This fact restricts the possible operator orderings to

$$\hat{\mathcal{H}} = -\frac{1}{2}\nabla^2 + \xi\mathcal{R} + U(q)\,, \tag{75}$$

where ∇^2 and $\mathcal{R}$ are the Laplacian and curvature of the minisuperspace metric $f_{\alpha\beta}$ and ξ is an arbitrary constant.

Before concluding this section, an important issue has to be addressed. It is how to interpret the probability measure in quantum cosmology. As a final remark we recall how we can interpret the probability measure. Since the Wheeler-DeWitt equation is similar to Klein-Gordon equation we can define a current that is conserved and satisfies $\nabla \cdot J = 0$. In this way

$$J = \frac{i}{2}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)\,. \tag{76}$$

As in the case of the Klein-Gordon equation (and, in general, of hyperbolic equations), the probability derived from such a conserved current can be affected by negative probabilities. Due to this shortcoming, the correct measure to use should be

$$dP = |\Psi(q^\alpha)|^2 dV\,, \tag{77}$$

where dV is a volume element of minisuperspace [6, 5].

3 The Noether-Symmetry Approach

As we said before, minisuperspaces are restrictions of the superspace of geometrodynamics. They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined. Cosmological models of physical interest can be defined on such minisuperspaces (*e.g.* Bianchi models). According to the above discussion, a crucial role is played by the conserved currents that allow one to interpret the probability measure and then the physical quantities obtained in quantum cosmology. In this context, the search for general methods to achieve conserved quantities and symmetries become relevant. The so-called *Noether Symmetry Approach* [4], as we will show, can be extremely useful to this purpose.

Let $\mathcal{L}$ a Lagrangian defined on the tangent space of configurations $T\mathcal{Q} \equiv \{q_i, \dot{q}_i\}$, the vector field X is

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial}{\partial \dot{q}^i}\,, \tag{78}$$

where dot means derivative with respect to t, and

$$L_X\mathcal{L} = X\mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,, \tag{79}$$

[19] Clearly, this is not the probability to select a given metric related with a given configuration of fields. If so, we would have fully solved the problem of quantum Gravity.

where $X\mathcal{L}$ is the Lie derivative L_X of $\mathcal{L}$. The condition

$$L_X\mathcal{L} = 0 \tag{80}$$

implies that the phase flux is conserved along X: this means that a constant of motion exists for $\mathcal{L}$ and Noether's theorem holds. In fact, taking into account the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \frac{\partial\mathcal{L}}{\partial q^i} = 0\,, \tag{81}$$

it is easy to show that

$$\frac{d}{dt}\left(\alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) = L_X\mathcal{L}\,. \tag{82}$$

If Eq.(80) holds,

$$\Sigma_0 = \alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i} \tag{83}$$

is a constant of motion. Alternatively, using the Cartan one–form

$$\theta_{\mathcal{L}} \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,dq^i \tag{84}$$

and defining the inner derivative

$$i_X\theta_{\mathcal{L}} = \langle\theta_{\mathcal{L}}, X\rangle\,, \tag{85}$$

we get, as above,

$$i_X\theta_{\mathcal{L}} = \Sigma_0\,, \tag{86}$$

if condition (80) holds. This representation is useful to identify cyclic variables. Using a point transformation on vector field (78), it is possible to get[20]

$$\tilde{X} = (i_X dQ^k)\,\frac{\partial}{\partial Q^k} + \left[\frac{d}{dt}(i_X dQ^k)\right]\frac{\partial}{\partial\dot{Q}^k}\,. \tag{87}$$

If X is a symmetry then also $\tilde{X}$ has this property, and it is always possible to choose a coordinate transformation so that

$$i_X dQ^1 = 1\,,\quad i_X dQ^i = 0\,,\quad i \neq 1\,, \tag{88}$$

and then

$$\tilde{X} = \frac{\partial}{\partial Q^1}\,,\quad \frac{\partial\tilde{\mathcal{L}}}{\partial Q^1} = 0\,. \tag{89}$$

It is evident that Q^1 is the cyclic coordinate and the dynamics can be reduced [8]. However, the change of coordinates is not unique and a clever choice is always important. Furthermore, it is possible that more symmetries are found. In this case, more than one cyclic variable exists.

A reduction procedure by cyclic coordinates can be implemented in three steps:

[20] We indicate the quantities as Lagrangians and vector fields with a tilde if the non–degenerate transformation

$$Q^i = Q^i(q)\,,\quad \dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j}\,\dot{q}^j\,,$$

is performed. However the Jacobian determinant $\mathcal{J} = \parallel \partial Q^i/\partial q^j \parallel$ has to be non–zero.

- we choose a symmetry and obtain new coordinates as above. After this first reduction, we get a new Lagrangian $\tilde{\mathcal{L}}$ with a cyclic coordinate;
- we search for new symmetries in this new space and apply the reduction technique until it is possible;
- the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic.

Going back to the point of view interesting in quantum cosmology, any symmetry selects a constant conjugate momentum since, by the Euler-Lagrange equations

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q^i} = 0 \Longleftrightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}^i} = \Sigma_i \,. \tag{90}$$

vice versa, the existence of a constant conjugate momentum means that a cyclic variable has to exist. In other words, a Noether symmetry exists.

Further remarks on the form of the Lagrangian $\mathcal{L}$ are necessary at this point. We shall take into account time–independent, non–degenerate Lagrangians $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^j)$, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial t} = 0\,, \quad \det H_{ij} \equiv \det || \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} || \neq 0\,, \tag{91}$$

where H_{ij} is the Hessian. As in usual analytic mechanics, $\mathcal{L}$ can be set in the form

$$\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i)\,, \tag{92}$$

where T is a positive–definite quadratic form in the $\dot{q}^j$ and $V(q^i)$ is a potential term. The energy function associated with $\mathcal{L}$ is

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}(q^j, \dot{q}^j)\,, \tag{93}$$

and by the Legendre transformation

$$\mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^j, \dot{q}^j)\,, \quad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{q}^j}\,, \tag{94}$$

we get the Hamiltonian function and the conjugate momenta. Considering again the symmetry, the condition (80) and the vector field X in Eq.(78) give a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the q^j. Due to (80), such a polynomial has to be identically zero and then each coefficient must be independently zero. If n is the dimension of the configuration space (*i.e.* the dimension of the minisuperspace), we get $\{1 + n(n+1)/2\}$ partial differential equations whose solutions assign the symmetry, as we shall see below. Such a symmetry is over–determined and, if a solution exists, it is expressed in terms of integration constants instead of boundary conditions. In the Hamiltonian formalism, we have

$$[\Sigma_j, \mathcal{H}] = 0\,, \quad 1 \leq j \leq m\,, \tag{95}$$

as it must be for conserved momenta in quantum mechanics, and the Hamiltonian has to satisfy the relations

$$L_\Gamma \mathcal{H} = 0 \tag{96}$$

in order to obtain a Noether symmetry. The vector Γ is defined by [9]

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \,. \tag{97}$$

These considerations can be applied to the minisuperspace models of quantum cosmology and to the interpretation of the wave function of the universe.

As discussed above, by a straightforward canonical quantization procedure, we have

$$\pi_j \longrightarrow \hat{\pi}_j = -i\partial_j \,, \tag{98}$$
$$\mathcal{H} \longrightarrow \hat{\mathcal{H}}(q^j, -i\partial_{q^j}) \,. \tag{99}$$

It is well known that the Hamiltonian constraint gives the Wheeler-DeWitt equation, so that if $|\Psi>$ is a *state* of the system (*i.e.* the wave function of the universe), dynamics is given by

$$\mathcal{H}|\Psi>= 0 \,, \tag{100}$$

where we write the Wheeler-DeWitt equation in an operational way. If a Noether symmetry exists, the reduction procedure outlined above can be applied and then, from (90) and (94), we get

$$\begin{aligned} \pi_1 \equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^1} = i_{X_1}\theta_{\mathcal{L}} &= \Sigma_1 \,, \\ \pi_2 \equiv \frac{\partial \mathcal{L}}{\partial \dot{Q}^2} = i_{X_2}\theta_{\mathcal{L}} &= \Sigma_2 \,, \\ \ldots \quad \ldots \quad &\quad \ldots \,, \end{aligned} \tag{101}$$

depending on the number of Noether symmetries. After quantization we get

$$\begin{aligned} -i\partial_1|\Psi> &= \Sigma_1|\Psi> \,, \\ -i\partial_2|\Psi> &= \Sigma_2|\Psi> \,, \\ \ldots \quad &\quad \ldots \,, \end{aligned} \tag{102}$$

which are nothing else but translations along the Q^j axis singled out by corresponding symmetry. Eqs. (102) can be immediately integrated and, Σ_j being real constants, we obtain oscillatory behaviors for $|\Psi>$ in the directions of symmetries, *i.e.*

$$|\Psi>= \sum_{j=1}^{m} e^{i\Sigma_j Q^j}|\chi(Q^l)> \,, \quad m < l \leq n \,, \tag{103}$$

where m is the number of symmetries, l are the directions where symmetries do not exist, and n is the total dimension of minisuperspace. Vice versa, dynamics given by (100) can be reduced by (102) if and only if it is possible to define constant conjugate momenta as in (101), that is, oscillatory behaviors of a subset of solutions $|\Psi>$ exist only if a Noether symmetry exists for dynamics.

The m symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two–dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of quantum cosmology [10]. In conclusion, we can state that in the semi-classical limit of quantum cosmology, the reduction procedure of dynamics, connected to the existence of Noether symmetries, allows one to select a subset of the solution of Wheeler-DeWitt equation where oscillatory behaviors are found. This fact, in the framework of the Hartle interpretative criterion of the wave function of the universe, gives conserved momenta and trajectories which can be interpreted as classical cosmological solutions. Vice versa, if a subset of the solution of Wheeler-DeWitt equation has an oscillatory behavior, due to Eq.(80), conserved momenta exist and Noether symmetries are present. In other words, *Noether symmetries select classical universes and then are directly related to the validity of the Hartle criterion*.

In what follows, we will show that such a statement holds for general classes of minisuperspaces and allows one to select exact classical solutions, *i.e.* the presence of Noether symmetries is a selection criterion for classical universes. In the next section, we will shown how they work for extended theories of gravitaty, as scalar tensor, $f(R)$ and $f(T)$ gravity.

4 Noether symmetries in cosmology

Let us consider realizations of the above approach in minisuperspace cosmological models derived from extended theories of gravity. Such theories of gravity have been recently invoked to address, for example, problems like dark energy and dark matter in cosmology and astrophysics [11]. As we have seen, the Hartle criterion is directly connected to the presence of Noether symmetries since oscillatory behaviors means correlations among variables [12, 13, 14, 11, 15].

Specifically, the approach can be connected to the search for Lagrange multipliers. In fact, imposing Lagrange multipliers allows one to modify the dynamics and select the form of effective potentials. By integrating the multipliers, solutions can be obtained.

On the other hand, the Lagrange multipliers are constraints capable of reducing dynamics. Technically they are anholonomic constraints being time-dependent. They give rise to field equations which describe dynamics of the further degrees of freedom coming from extended theories of gravity [12, 13, 14, 11, 15]. This fact is extremely relevant to deal with new degrees of freedom under the standard of effective scalar fields [16]. Below, we give minisuperspace examples and obtain exact cosmological solutions. In particular, we show that, by imposing Lagrange multipliers, a given minisuperspace model becomes canonical and Noether symmetries, if exist, can be found out.

4.1 Scalar-Tensor cosmology

A general action for a nonminimally coupled theory of gravity assumes the following form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - V(\phi)\right] , \tag{104}$$

where, $F(\phi)$ and $V(\phi)$ are respectively the coupling and the potential of a scalar field [21].

[21]Using physical units $8\pi G = c = \hbar = 1$, the standard Einstein coupling is recovered for $F(\phi) = -1/2$.

The cosmological point-like Lagrangian for a Friedman Robertson Walker (FRW) minisuperspace in terms of the scale factor a is

$$\mathcal{L} = 6a\dot{a}^2F + 6a^2\dot{a}\dot{F} - 6kaF + a^3\left[\frac{\dot{\phi}}{2} - V\right]\,. \tag{105}$$

The configuration space of such a Lagrangian is $\mathcal{Q} \equiv \{a, \phi\}$, *i.e.* a bidimensional minisuperspace. A Noether symmetry exists if Eq. (80) holds. In this case, it has to be

$$X = \alpha\,\frac{\partial}{\partial a} + \beta\,\frac{\partial}{\partial \phi} + \dot{\alpha}\,\frac{\partial}{\partial \dot{a}} + \dot{\beta}\,\frac{\partial}{\partial \dot{\phi}}\,, \tag{106}$$

where α, β depend on a, ϕ. This vector field acts on the $\mathcal{Q}$ minisuperspace. The system of partial differential equation given by (80) is

$$a \qquad F(\phi)\left[\alpha + 2a\frac{\partial\alpha}{\partial a}\right] + aF'(\phi)\left[\beta + a\frac{\partial\beta}{\partial a}\right] = 0\,, \tag{107}$$

$$3\alpha + 12F'(\phi)\frac{\partial\alpha}{\partial\phi} + 2a\frac{\partial\beta}{\partial\phi} = 0\,, \tag{108}$$

$$a\beta F''(\phi) + \left[2\alpha + a\frac{\partial\alpha}{\partial a} + \frac{\partial\beta}{\partial\phi}\right]F'(\phi) + 2\frac{\partial\alpha}{\partial\phi}F(\phi) + \frac{a^2}{6}\frac{\partial\beta}{\partial a} = 0\,, \tag{109}$$

$$[3\alpha V(\phi) + a\beta V'(\phi)]a^2 + 6k[\alpha F(\phi) + a\beta F'(\phi)] = 0\,. \tag{110}$$

Prime indicates the derivative with respect to ϕ. The number of equations is 4 as it has to be, the $\mathcal{Q}$-dimension being $n = 2$. Several solutions exist for this system [17, 18, 19]. They determine also the form of the model since the system (107)-(110) gives α, β, $F(\phi)$ and $V(\phi)$. For example, if the spatial curvature is $k = 0$, a solution is

$$\alpha = -\frac{2}{3}p(s)\beta_0 a^{s+1}\phi^{m(s)-1}\,, \quad \beta = \beta_0 a^s\phi^{m(s)}\,, \tag{111}$$

$$F(\phi) = D(s)\phi^2\,, \quad V(\phi) = \lambda\phi^{2p(s)}\,, \tag{112}$$

where

$$D(s) = \frac{(2s+3)^2}{48(s+1)(s+2)}\,, \tag{113}$$

$$p(s) = \frac{3(s+1)}{2s+3}\,, \tag{114}$$

$$m(s) = \frac{2s^2+6s+3}{2s+3}\,, \tag{115}$$

and s, λ are free parameters. The change of variables (88) gives

$$w = \sigma_0 a^3 \phi^{2p(s)}\,, \quad z = \frac{3}{\beta_0 \chi(s)} a^{-s} \phi^{1-m(s)}\,, \tag{116}$$

where σ_0 is an integration constant and

$$\chi(s) = -\frac{6s}{2s+3}\,. \tag{117}$$

The Lagrangian (105) becomes, for $k = 0$,

$$\mathcal{L} = \gamma(s) w^{s/3} \dot{z}\dot{w} - \lambda w\,, \tag{118}$$

where z is cyclic and

$$\gamma(s) = \frac{2s+3}{12\sigma_0^2 (s+2)(s+1)}\,. \tag{119}$$

The conjugate momenta are

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s) w^{s/3} \dot{w}\,, \quad \pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s) w^{s/3} \dot{z}\,, \tag{120}$$

and the Hamiltonian is

$$\tilde{\mathcal{H}} = \frac{\pi_z \pi_w}{\gamma(s) w^{s/3}} + \lambda w\,. \tag{121}$$

The Noether symmetry is given by

$$\pi_z = \Sigma_0\,. \tag{122}$$

Quantizing Eqs. (120), we get

$$\pi \longrightarrow -i\partial_z\,, \quad \pi_w \longrightarrow -i\partial_w\,, \tag{123}$$

and then the Wheeler-DeWitt equation

$$[(i\partial_z)(i\partial_w) + \tilde{\lambda} w^{1+s/3}]|\Psi>= 0\,, \tag{124}$$

where $\tilde{\lambda} = \gamma(s)\lambda$. The quantum version of constraint (122) is

$$-i\partial_z|\Psi>= \Sigma_0|\Psi>\;, \tag{125}$$

so that the resulting dynamics is reduced. A straightforward integration of Eqs. (124) and (125) gives

$$|\Psi>= |\Omega(w)>|\chi(z)>\propto e^{i\Sigma_0 z}\, e^{-i\tilde{\lambda} w^{2+s/3}}\,, \tag{126}$$

which is an oscillating wave function, and the Hartle criterion is recovered. In the semi–classical limit, we have two first integrals of motion: Σ_0 (*i.e.* the equation for π_z) and $E_{\mathcal{L}} = 0$,*i.e.* the Hamiltonian (121) which becomes the equation for π_w. Classical trajectories in the configuration space $\tilde{\mathcal{Q}} \equiv \{w, z\}$ are immediately recovered

$$w(t) = [k_1 t + k_2]^{3/(s+3)}\,, \tag{127}$$

$$z(t) = [k_1 t + k_2]^{(s+6)/(s+3)} + z_0\,, \tag{128}$$

then, going back to $\mathcal{Q} \equiv \{a, \phi\}$, we get the *classical* cosmological behaviour

$$a(t) = a_0(t - t_0)^{l(s)}, \quad (129)$$

$$\phi(t) = \phi_0(t - t_0)^{q(s)}, \quad (130)$$

where

$$l(s) = \frac{2s^2 + 9s + 6}{s(s+3)}, \quad q(s) = -\frac{2s+3}{s}, \quad (131)$$

which means that Hartle criterion selects classical universes. Depending on the value of s, we get Friedman, power–law, or pole–like behaviors. The considerations on the oscillatory regime of the wave function of the universe and the recovering of classical behaviors are exactly the same.

4.2 $f(R)$ cosmology

Similar results can be obtained also for higher–order gravity minisuperspaces. In particular, let us consider fourth–order gravity given by the action

$$\mathcal{S} = \int d^4x \sqrt{-g}\, f(R), \quad (132)$$

where $f(R)$ is a generic function of Ricci scalar curvature [11]. Rewriting the action to a point-like Friedman-Robertson-Walker one, we obtain

$$\mathcal{S} = \int dt \mathcal{L}(a, \dot{a}; R, \dot{R}), \quad (133)$$

where dot means derivative with respect to the cosmic time. The scale factor a and the Ricci scalar R are the canonical variables. This position could seem arbitrary since R depends on $a, \dot{a}, \ddot{a}$, but it is generally used in canonical quantization [20, 21, 22]. The definition of R in terms of $a, \dot{a}, \ddot{a}$ introduces a constraint which eliminates second and higher order derivatives in action (133), and yields to a system of second order differential equations in $\{a, R\}$. Action (133) can be written as

$$\mathcal{S} = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[R + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}, \quad (134)$$

where the Lagrange multiplier λ is derived by varying with respect to R. It is

$$\lambda = a^3 f'(R). \quad (135)$$

Here prime means derivative with respect to R. To recover the analogy with previous scalar–tensor models, let us introduce the auxiliary field

$$p \equiv f'(R), \quad (136)$$

so that the Lagrangian in (134) becomes

$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2\dot{a}\dot{p} - 6kap - a^3 W(p), \quad (137)$$

which is of the same form of (105) a part the kinetic term. This is a Helmholtz–like Lagrangian [23] and a, p are independent fields. The potential $W(p)$ is defined as

$$W(p) = h(p)p - r(p)\,, \tag{138}$$

where

$$r(p) = \int f'(R)dR = \int pdR = f(R)\,, \quad h(p) = R\,, \tag{139}$$

such that $h = (f')^{-1}$ is the inverse function of f'. The configuration space is now $\mathcal{Q} \equiv \{a, p\}$ and p has the same role of the above ϕ. Condition (80) is now realized by the vector field

$$X = \alpha(a,p)\frac{\partial}{\partial a} + \beta(a,p)\frac{\partial}{\partial p} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{p}}\,, \tag{140}$$

and explicitly it gives the system

$$a \qquad p\left[\alpha + 2a\frac{\partial\alpha}{\partial a}\right]p + a\left[\beta + a\frac{\partial\beta}{\partial a}\right] = 0\,, \tag{141}$$

$$a^2\frac{\partial\alpha}{\partial p} = 0\,, \tag{142}$$

$$2\alpha + a\frac{\partial\alpha}{\partial a} + 2p\frac{\partial\alpha}{\partial p} + a\frac{\partial\beta}{\partial p} = 0\,, \tag{143}$$

$$6k[\alpha p + \beta a] + a^2[3\alpha W + a\beta\frac{\partial W}{\partial p}] = 0\,. \tag{144}$$

The solution of this system, *i.e.* the existence of a Noether symmetry, gives α, β and $W(p)$. It is satisfied for

$$\alpha = \alpha(a)\,, \qquad \beta(a,p) = \beta_0 a^s p\,, \tag{145}$$

where s is a parameter and β_0 is an integration constant. In particular,

$$s = 0 \longrightarrow \alpha(a) = -\frac{\beta_0}{3}\,a\,, \quad \beta(p) = \beta_0\,p\,,$$

$$W(p) = W_0\,p\,, \quad k = 0\,, \tag{146}$$

$$s = -2 \longrightarrow \alpha(a) = -\frac{\beta_0}{a}\,, \quad \beta(a,p) = \beta_0\,\frac{p}{a^2}\,,$$

$$W(p) = W_1 p^3\,, \quad \forall\ k\,, \tag{147}$$

where W_0 and W_1 are constants. Let us discuss separately the solutions (146) and (147).

4.3 The case $s = 0$

The induced change of variables $\mathcal{Q} \equiv \{a, p\} \longrightarrow \tilde{Q} \equiv \{w, z\}$ can be

$$w(a,p) = a^3 p\,, \quad z(p) = \ln p\,. \tag{148}$$

Lagrangian (137) becomes

$$\tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = \dot{z}\dot{w} - 2w\dot{z}^2 + \frac{\dot{w}^2}{w} - 3W_0 w\,. \tag{149}$$

and, obviously, z is the cyclic variable. The conjugate momenta are

$$\pi_z \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = \dot{w} - 4\dot{z} = \Sigma_0\,, \tag{150}$$

$$\pi_w \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = \dot{z} + 2\frac{\dot{w}}{w}\,. \tag{151}$$

and the Hamiltonian is

$$\mathcal{H}(w, \pi_w, \pi_z) = \pi_w \pi_z - \frac{\pi_z^2}{w} + 2w\pi_w^2 + 6W_0 w\,. \tag{152}$$

By canonical quantization, reduced dynamics is given by

$$\left[\partial_z^2 - 2w^2\partial_w^2 - w\partial_w\partial_z + 6W_0 w^2\right] |\Psi >= 0\,, \tag{153}$$

$$- i\partial_z|\Psi >= \Sigma_0\,|\Psi > \ . \tag{154}$$

However, we have done simple factor ordering considerations in the Wheeler-DeWitt equation (153). Immediately, the wave function has an oscillatory factor, being

$$|\Psi > \sim e^{i\Sigma_0 z}|\chi(w) > \ . \tag{155}$$

The function $|\chi >$ satisfies the Bessel differential equation

$$\left[w^2\partial_w^2 + i\frac{\Sigma_0}{2}\, w\,\partial_w + \left(\frac{\Sigma_0^2}{2} - 3W_0 w^2\right)\right]\chi(w) = 0\,, \tag{156}$$

whose solutions are linear combinations of Bessel functions $Z_\nu(w)$

$$\chi(w) = w^{1/2 - i\Sigma_0/4} Z_\nu(\lambda w)\,, \tag{157}$$

where

$$\nu = \pm\frac{1}{4}\sqrt{4 - 9\Sigma_0^2 - i4\Sigma_0}\,, \quad \lambda = \pm 9\sqrt{\frac{W_0}{2}}\,. \tag{158}$$

The oscillatory regime for this component depends on the reality of ν and λ. The wave function of the universe, from Noether's symmetry (146) is then

$$\Psi(z, w) \sim e^{i\Sigma_0[z - (1/4)\ln w]}\, w^{1/2} Z_\nu(\lambda w)\,. \tag{159}$$

For large w, the Bessel functions have an exponential behavior, so that the wave function (159) can be written as

$$\Psi \sim e^{i[\Sigma_0 z - (\Sigma_0/4)\ln w \pm \lambda w]}\,. \tag{160}$$

Due to the oscillatory behaviour of Ψ, Hartle's criterion is immediately recovered. By identifying the exponential factor of (160) with S_0, we can recover the conserved momenta π_z, π_w and select classical trajectories. Going back to the old variables, we get the cosmological solutions

$$a(t) = a_0 e^{(\lambda/6)t}\,\exp\left\{-\frac{z_1}{3}\,e^{-(2\lambda/3)t}\right\}, \tag{161}$$

$$p(t) = p_0 e^{(\lambda/6)t}\,\exp\left\{z_1\,e^{-(2\lambda/3)t}\right\}, \tag{162}$$

where a_0, p_0 and z_1 are integration constants. It is clear that λ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered.

4.4 The case $s = -2$

The new variables adapted to the foliation for the solution (147) are now

$$w(a,p) = ap\,, \qquad z(a) = a^2\,. \tag{163}$$

and Lagrangian (137) assumes the form

$$\tilde{\mathcal{L}}(w,\dot{w},\dot{z}) = 3\dot{z}\dot{w} - 6kw - W_1 w^3. \tag{164}$$

The conjugate momenta are

$$\pi_z = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{z}} = 3\dot{w} = \Sigma_1\,, \tag{165}$$

$$\pi_w = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{w}} = 3\dot{z}\,. \tag{166}$$

The Hamiltonian is given by

$$\mathcal{H}(w,\pi_w,\pi_z) = \frac{1}{3}\,\pi_z\pi_w + 6kw + W_1 w^3\,. \tag{167}$$

Going over the same steps as above, the wave function of the universe is given by

$$\Psi(z,w) \sim e^{i[\Sigma_1 z + 9kw^2 + (3W_1/4)w^4]}\,, \tag{168}$$

and the classical cosmological solutions are

$$a(t) = \pm\sqrt{h(t)}\,, \qquad p(t) = \pm\frac{c_1 + (\Sigma_1/3)\,t}{\sqrt{h(t)}}\,, \tag{169}$$

where

$$h(t) = \left(\frac{W_1\Sigma_1^3}{36}\right)t^4 + \left(\frac{W_1 w_1\Sigma_1}{6}\right)t^3 + \left(k\Sigma_1 + \frac{W_1 w_1^2\Sigma_1}{2}\right)t^2 + w_1(6k + W_1 w_1^2)\,t + z_2\,. \tag{170}$$

w_1, z_1 and z_2 are integration constants. Immediately we see that for large t,

$$a(t) \sim t^2\,, \qquad p(t) \sim \frac{1}{t}, \tag{171}$$

which is a power-law inflationary behavior. An extensive discussion of Noether symmetries in $f(R)$ gravity is in [24, 25].

4.5 $f(T)$ cosmology

In analogy to the $f(R)$ gravity, a new sort of extended gravity, the so-called $f(T)$ theory, has been recently proposed. It is a generalized and extended version of the teleparallel gravity originally proposed by Einstein [26, 27, 28]. Teleparallelism uses as dynamical object a vierbein field $e_i(x^\mu)$, $i = 0,1,2,3$, which is an orthonormal basis for the tangent space at each point x^μ of the manifold: $e_i.e_j = \eta_{ij}$, where $\eta_{ij} = diag(-1,1,1,1)$. Each vector e_i can be described by its components e_i^μ, $\mu = 0,1,2,3$ in a coordinate basis; i.e. $e_i = e_i^\mu \partial_\mu$. Notice that Latin indexes refer to the tangent space, while Greek indexes label coordinates on the manifold. The metric tensor is obtained from the dual vierbein as $g_{\mu\nu}(x) = \eta_{ij} e^i_\mu(x) e^j_\nu(x)$. Furthermore the Weitzenböck connection is used, whose non-null torsion is

$$T^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\nu\mu} - \hat{\Gamma}^\lambda_{\mu\nu} = e^\lambda_i(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu), \tag{172}$$

rather than the Levi-Civita connection which is used in general relativity.

This tensor encompasses all the information about the gravitational field. The Lagrangian is built by the torsion (172) and its dynamical equations for the vierbein imply the Einstein equations for the metric. The teleparallel Lagrangian is

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \tag{173}$$

where

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta) \tag{174}$$

and $K^{\mu\nu}{}_\rho$ is the contorsion tensor

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}), \tag{175}$$

which equals the difference between Weitzenböck and Levi-Civita connections. Thus the general action assume the following form

$$\mathcal{S} = \int d^4x e f(T), \tag{176}$$

where $e = det(e^i_\mu) = \sqrt{-g}$. If matter couples to the metric in the standard form then the variation of the action with respect to the vierbein leads to the equations [29]

$$e^{-1}\partial_\mu(e S_i{}^{\mu\nu}) f'(T) - e^\lambda_i T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} f'(T) + S_i{}^{\mu\nu}\partial_\mu(T) f''(T) + \frac{1}{4} e^\nu_i f(T) = e_i{}^\rho T_\rho{}^\nu, \tag{177}$$

where a prime denotes differentiation with respect to T, $S_i^{\mu\nu} = e_i^{\rho} S_{\rho}^{\mu\nu}$ and $T_{\mu\nu}$ is the matter energy-momentum tensor. In order to derive the cosmological equations in a Friedmann-Robertson-Walker (FRW) metric, we need to infer, as above, a point-like Lagrangian from the action(176). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. Then we can write

$$e^i_\mu = diag(1, a(t), a(t), a(t)), \tag{178}$$

where $a(t)$ is the cosmological scale factor, and the action is

$$\mathcal{S} = \int \mathcal{L}(a, \dot{a}, T, \dot{T}) dt \tag{179}$$

considering a and T as canonical variables, whereas $\mathcal{Q} = \{a, T\}$ is the configuration space, and $\mathcal{TQ} = \{a, \dot{a}, T, \dot{T}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. As above, one can use the method of Lagrange multipliers to set T as a constraint of the dynamics. Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical and then we have [24, 30, 31, 32]

$$\mathcal{S} = 2\pi^2 \int dt\, a^3 \left[f(T) - \lambda \left(T + 6\frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right], \tag{180}$$

where λ is a Lagrange multiplier. The variation with respect to T gives

$$\lambda = f'(T) \,. \tag{181}$$

Therefore, the action (180) can be rewritten as

$$\mathcal{S} = 2\pi^2 \int dt\, a^3 \left[f(T) - f'(T) \left(T + 6\frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right], \tag{182}$$

and then the point-like Lagrangian reads

$$\mathcal{L}(a, \dot{a}, T, \dot{T}) = a^3 \left[f(T) - f'(T)T \right] - 6f'(T)a\dot{a}^2 - \rho_{m0} \,. \tag{183}$$

Substituting Eq. (183) into the Euler-Lagrange equation, we obtain

$$a^3 f''(T) \left(T + 6\frac{\dot{a}^2}{a^2} \right) = 0 \,, \tag{184}$$

$$f(T) - f'(T)T + 2f_T H^2 + 4\left[f'(T)\frac{\ddot{a}}{a} + H f''(T)\dot{T} \right] = 0 \,. \tag{185}$$

If $f''(T) \neq 0$, from Eq. (184), it is easy to find that

$$T = -6\left(\frac{\dot{a}}{a}\right)^2 = -6H^2 \,. \tag{186}$$

This is the Euler constraint for dynamics. Substituting Eq. (186) into Eq. (185) and using $\ddot{a}/a = H^2 + \dot{H}$, we get

$$48H^2 f''(T)\dot{H} - 4f'(T)\left(3H^2 + \dot{H}\right) - f(T) = 0\,. \tag{187}$$

By a Legendre transformation on Lagrangian (183), we obtain the Hamiltonian constraint

$$\mathcal{H}(a, \dot{a}, T, \dot{T}) = a^3\left[-6f'(T)\frac{\dot{a}^2}{a^2} - f(T) + f'(T)T + \frac{\rho_{m0}}{a^3}\right]. \tag{188}$$

Considering the total energy $\mathcal{H} = 0$ [33, 24, 32] and using Eq. (186), we get

$$12H^2 f'(T) + f(T) = \frac{\rho_{m0}}{a^3}\,. \tag{189}$$

In summary, the point-like Lagrangian(183) yields all the equations of motion [34]. To obtain the Noether symmetries, we substitute Eq. (183) into Eq. (79) imposing $L_X\mathcal{L} = 0$ and using the relations $\dot{\alpha} = (\partial\alpha/\partial a)\,\dot{a} + (\partial\alpha/\partial T)\,\dot{T}$, $\dot{\beta} = (\partial\beta/\partial a)\,\dot{a} + (\partial\beta/\partial T)\,\dot{T}$, we obtain

$$\begin{aligned}
&3\alpha a^2\left[f(T) - f'(T)T\right] - \beta a^3 f''(T)T + \\
&-6\dot{a}^2\left[\alpha f'(T) + \beta a f''(T) + 2af'(T)\frac{\partial\alpha}{\partial a}\right] - 12a\dot{a}\dot{T}\frac{\partial\alpha}{\partial T} = 0\,.
\end{aligned} \tag{190}$$

As mentioned above, requiring the coefficients of $\dot{a}^2$, $\dot{T}^2$ and $\dot{a}\dot{T}$ in Eq. (190) to be zero, we find that

$$a\frac{\partial\alpha}{\partial T} = 0\,, \tag{191}$$

$$\alpha f'(T) + \beta a f''(T) + 2af'(T)\frac{\partial\alpha}{\partial a} = 0\,, \tag{192}$$

$$3\alpha a^2\left(f(T) - f'(T)T\right) - \beta a^3 f''(T)T = 0\,. \tag{193}$$

In particular, the constraint (193) is the *Noether condition* [24, 34]. The corresponding constant of motion (*Noether's charge*), reads

$$Q_0 = -12\alpha f'(T)a\dot{a} = const. \tag{194}$$

A solution of Eqs. (191), (192) and (193) exists if explicit forms of α and β are found. In this case, as above, a symmetry exists. Obviously, from Eq. (191), it is easy to see that α is independent of T, and hence it is a function of a only, *i.e.*, $\alpha = \alpha(a)$. On the other hand, from Eq. (193), we have

$$\beta a f''(T)T = 3\alpha\left(f(T) - f'(T)T\right). \tag{195}$$

Multiplying by T Eq. (192), and then substituting Eq. (195) into it, we obtain

$$f'(T)T\left(2a\frac{d\alpha}{da} - 2\alpha\right) + 3\alpha f(T) = 0\,. \tag{196}$$

One can perform a separation of variables and recast Eq. (196) as

$$1 - \frac{a}{\alpha}\frac{d\alpha}{da} = \frac{3f(T)}{2f'(T)T} \,. \tag{197}$$

Since its left-hand side is a function of a only and its right-hand side is a function of T only, they must be equal to a constant in order to ensure that Eq. (197) holds. For convenience, we let this constant be $3/(2n)$, and then Eq. (197) can be separated into two ordinary differential equations, *i.e.*,

$$nf(T) = f'(T)T \,, \qquad 1 - \frac{a}{\alpha}\frac{d\alpha}{da} = \frac{3}{2n} \,. \tag{198}$$

It is easy to find the solutions of these two ordinary differential equations, namely

$$f(T) = \mu_0 T^n \,, \qquad \alpha(a) = \alpha_0 \, a^{1-3/(2n)} \,, \tag{199}$$

where μ_0 and α_0 are integral constants. Obviously, $f(T)$ and $\alpha(a)$ are both power-law functions. Substituting Eqs. (198) and (199) into Eq. (195), we find that

$$\beta(a,T) = -\frac{3\alpha_0}{n} \, a^{-3/(2n)} \, T \,. \tag{200}$$

Therefore a Noether symmetry exists. Finally, we find out the exact solution $a(t)$ for this family of $f(T)$. Substituting Eqs. (198), (199) and (186) into Eq. (194), we obtain an ordinary differential equation for $a(t)$, namely

$$a^{c_1} \, \dot{a} = c_2 \,, \tag{201}$$

where

$$c_1 = \frac{3}{2n} - 1 \,, \qquad c_2 = \left[\frac{Q_0}{-12\alpha_0\mu_0 n(-6)^{n-1}}\right]^{1/(2n-1)} \,. \tag{202}$$

It is easy to find that the general solution for Eq. (201) is

$$a(t) = -(1+c_1)(c_3 - c_2 t)^{1/(1+c_1)} = (-1)^{1+2n/3} \cdot \frac{3}{2n} \, (c_2 t - c_3)^{2n/3} \,, \tag{203}$$

where c_3 is another integration constant. Obviously, in the late time $|c_2 t| \gg |c_3|$ and the universe experiences a power-law expansion. Requiring $a(t=0) = 0$, it is easy to see that the integration constant c_3 is zero. Finally, we have a behavior of the form

$$a(t) \sim t^{2n/3} \,, \tag{204}$$

which is clearly a Friedman behavior. Note that the condition $n > 0$ is required to ensure that the universe is expanding [34].

5 Discussion and Conclusions

In this paper, we discussed Noether symmetries in cosmology. The approach consists in identifying conserved quantities that select peaked behaviors in the solutions of the Wheeler-DeWitt equation and then in solving exactly the related cosmological models. Peaked behaviors mean correlations among variables and then the possibility to obtain classical and observable universes according to the interpretative Hartle criterion of the wave function of the universe. Specifically, such a criterion states that classical observable universes are solutions of dynamics as soon as correlations among physical variables are identified. Here, we searched for Noether symmetries that allow one to reduce dynamics coming from minisuperspaces and then to find out exact solutions.

The method has been worked out for some extended theories of gravity, namely scalar-tensor gravity, $f(R)$ gravity and $f(T)$ gravity. The existence of symmetries fixes, in particular the form of functions like $f(R)$, $f(T)$, and so on. The common feature of such dynamical systems is that, in any case, specific Lagrange multipliers, related to symmetries, can be found out. Such multipliers allow one to reduce dynamics and then exact cosmological solutions can be found.

Here we have worked out some simple models but more physically interesting systems can be taken into account and compared to cosmological observations [24, 31, 25, 30]. In conclusion, beside integrability, the physical meaning of the models is fixed by symmetries that can be considered a sort of natural selection rule.

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Treasurer's report

IAMP has 683 members; the membership breakup is described in the table below. Due to currently low interests rates, the association's main source of annual income is the dues. Over the last 3 years, we welcomed about 100 new members. At the same time, IAMP lost roughly 150 other members, some through natural attrition and others due to prolonged lack of dues payment.

We happily welcomed SISSA (Italy) as a new associate member.

Membership category	Current Number	In 2015	In 2012
Registered	683	750	681
Ordinary (O)	370	428	367
In good standing	303	423	163
In payment arrears	23	5	34
Lifetime (L)	273	269	266
Reduced fee (R)	40	40	48
Associate (A)	14	13	N/A

IAMP continues to receive an annual contribution from the Daniel Iagolnitzer Foundation (DIF). The funds are earmarked for the IAMP Henri Poincaré Prize, sponsored by DIF, awarded triennially at the International Congress in Mathematical Physics (ICMP) as well as related expenses, and ICMP support. As the European accounts were consolidated into one, held in Bielefeld, the DIF funds are now sent annually there.

In a new contract, signed with Springer Publishing on the occasion of the 50th anniversary of *Communications in Mathematical Physics*, it became the official sponsor of IAMP's Early Career Award (ECA). The ECA prize continues to be awarded at the triennial ICMP. Springer has committed to provide 5,000 EUR per year (starting 2017). The funds are split between sponsorship of the ECA, which carries a prize of 3000 EUR, and contributions to conferences selected by IAMP. In the year of the ICMP, part of this money is earmarked for the travel support of the recipient of the ECA to the congress.

IAMP's main financial expenditure continues to be conference support. It has been a deliberate choice of the past Executive Committee (EC) to reduce IAMP's assets through our conference sponsoring programme. It provids small, but unbureaucratic funds. Aside from the ICMP, IAMP supports conferences in mathematical physics either through an official endorsement of the EC or, additionally, with up to 3,000 EUR seed funds.

Conference spending used to run up to 10 KEUR per year. In the past three years, thanks to many excellent conferences organized by the community and the additional funds from Springer, conference spending went up to 18 KEUR per year. A detailed overview of the conferences funded by IAMP in the period August 2015 – July 2018 can be found below. What is not shown there is that IAMP is already committed to spending 7,000 EUR on conferences in the period Aug 2018 – Dec 2018. Conference support at the current level is above the sustainable level which would be about 14 KEUR per year. It will be the policy of the new EC to reduce the amount of IAMP's contribution to each conference in order to retain the ability to support many conferences.

Financial activity 2015-2018	(EUR)	(USD)
Dues	43,458	3,357
Donations	3,058	52
Interest	22	24
Daniel Iagolnitzer Foundation	66,274	N/A
Springer ('17-'18)	10,000	N/A
Total Bank/CC fees	-3,283	-227
Internet Hosting	-205	N/A
Conference support	-40,164	-19,275
ECA Prize '15	-3,135	N/A
HPP/ICMP15 Support	-8,124	-3,497
HPP/ICMP18 Support	-55,000	0
Total Gain/Loss	12,901	-19,566

IAMP's expenses concerning the credit card system and the banking fees remained at approximately the same level, of about 1,000 EUR per year, as in the previous reporting period. IAMP's webpage is now hosted at 1&1, a professional service which charges 84 EUR per year.

The table below summarizes the development of IAMP's total assets. These are being held in checking and savings accounts in Bielefeld (Germany) and Birmingham (USA). As can be inferred from the table of financial activities, the EUR equivalent of IAMP's total losses during the last 3 years was about 4,000 EUR.

As is shown there, the total loss in the period 2015–2018 adds up to roughly 9,000 EUR. The discrepancy is caused by the depreciation of the US Dollar in comparison to the Euro.

IAMP continues to receive support in the form of services at no costs thanks to the generosity of some members. The Bulletin is edited by Evans Harrell. Günter Stolz takes care of the US accounts in Birmingham. Many thanks go also to my former secretary, Frauke Bäcker, who helped me with the bookkeeping of the finances until the end of last year, and to my current secretary, Mariola Stasch, who took over in January.

Despite of his retirement from the TU Braunschweig, Dietmar Kähler still maintains the membership database in Braunschweig. As some of you already witnessed when paying by credit card, the database's old software now produces occasional glitches. One of the major tasks of the new EC in the coming three years will be to renew the membership database and set up a solution which will last another 10 years.

Account	Balance	Currency	Euro equivalent
Bielefeld Checking	17,701	EUR	17,701
Bielefeld Savings	86,000	EUR	86,000
US Checking	2,215	USD	1,895
US Savings	18.666	USD	15,972
TOTAL Checking:			19,596
TOTAL Savings:			101,972
TOTAL (EUR):			121,568
As of Jun 30, 2003:			116,000
As of Jun 30, 2006:			112,598
As of Jun 30, 2009:			96,963
As of Jun 30, 2012:			125,545
As of Jun 30, 2015:			130,978

Overall, the treasury's main goal will be to maintaining the net yearly income from membership dues and donations at about 13,000 EUR. One of the issues IAMP will face is that the number of no-dues paying lifetime members will increase substantially by 2020 at a rate of roughly 40 members per year. Lifetime membership is available to members who have paid a total of 25 year dues on or after Jan 1, 1995. The above goal will therefore only be met by encouraging more donations and reaching out to nonIAMP members in the mathematical physics community. IAMP will need your help in supporting this!

Yours truly,
Simone Warzel, IAMP treasurer.

Conference [Location] (Aug 2015 – Jul 2018)	Amount
Stochastic & Analytic Methods in Mathematical Physics [AM] (2015)	2,000 EUR
New directions in statistical mechanics and dynamics systems [US] (2015)	1,700 USD
Mathematical Challenges in Quantum Mechanics [IT] (2016)	3,000 EUR
B. Simon conference CRM [CA] (2016)	3,366 USD
Geometric aspects of spectral theory [ES] (2016)	1,000 EUR
Mathematical many-body theory and its applications [ES] (2016)	1,500 EUR
Analysis and beyond [US] (2016)	2,170 USD
Great Lakes Mathematical Physics meeting [US] (2016)	1,500 USD
Summer school in Sirince [TUR] (2016)	1,422 USD
Mathematical Physics Days [DE] (2016)	1,000 EUR
Quantum Roundabout [UK] (2016)	1,000 EUR
Random Geometry and Physics [FR] (2016)	2,000 EUR
EMS/IAMP summer school [IT] (2016)	3,000 EUR
QMATH13 [US] (2016)	1,942 EUR
Spectral Days [DE] (2017)	1,800 EUR
Quantum Theory and Symmetries [BG] (2017)	2,400 EUR
Geometry and Relativity at ESI [AT] (2017)	2,000 EUR
Summer school in Probability at PIMS [CA] (2017)	2,104 USD
5th Quantum Thermodynamics conference [UK] (2017)	1,800 EUR
Mathematical Congress of America [CA] (2017)	2,117 USD
Mathematical aspects of the physics with non-self-adjoint op.s. [FR] (2017)	1,000 EUR
Mathematical Physics: from field theory to non-equilibrium [FR] (2017)	1,000 EUR
Quantissima in the Serenissima II [IT] (2017)	1,000 EUR
Probabilistic approaches in Mathematical Physics [ES] (2017)	1,000 EUR
A random event in honour of Ilya Goldsheid [UK] (2017)	1,500 EUR
Asymptotic analysis and spectral theory [DE] (2017)	1,000 EUR
Mathematical Physics Perspective of Billiards and Dominoes [US] (2017)	1,000 USD
ICMP Scientific Committee Meeting [CA] (2017)	722 EUR
	+ 1,420 USD
Mathematical Challenges in Quantum Mechanics [IT] (2018)	3,000 EUR
EMS-IAMP summer school [IT] (2018)	3,000 EUR
Current Topics in Mathematical Physics [CA] (2018)	2,476 USD
Quantum Roundabout [UK] (2018)	500 EUR
Total (2015-2018):	40,164 EUR
	+ 19,275 USD

News from the IAMP Executive Committee

New individual members

IAMP welcomes the following new members

1. DR. ALESSANDRO OLGIATI, SISSA, Trieste, Italy
2. PROF. FRÉDÉRIC HÉRAU, Nantes, France
3. DR. SERGIO SIMONELLA, ENS Lyon, France
4. DR. ANDREA GIUSTI, Bologna, Italy
5. PROF. VALERIO LUCARINI, Reading, UK
6. MS. CHRISTINA POSPISIL, University of Saarland, Germany
7. PROF. STEPHANE NONNENMACHER, Université Paris-Sud, France
8. MS. CRISTINA CARACI, GSSI L' Aquila, Italy
9. DR. FRANCESCA ARICI, MPI Leipzig, Germany

Next ICMP

The next International Congress of Mathematical Physics will be held in Geneva (Switzerland), from August 2 until August 7, 2021. It will be preceded by the Young Researcher Symposium, from July 29 to July 31, 2021.

Recent conference announcements

Master Class 2019/2020 in Mathematical Physics

Sept. 2019 - May 2020 (one-year program). University of Geneva, Switzerland. Organized by SwissMAP, aimed at Master and beginning PhD students. Deadline for applications: January 15, 2019.

http://www.nccr-swissmap.ch/master-class-2019

3-rd Bangkok Workshop on Discrete Geometry, Dynamics and Statistics

Jan. 21-25, 2019. Chulalongkorn University, Thailand.

http://thaihep.phys.sc.chula.ac.th/BKK2019DSCR/

Scaling limits and SPDEs: recent developments and future directions

Dec. 10-14, 2018. Newton Institute, Cambridge, UK.
Organized by A. Kupiainen, P. Goncalves, F. Otto, J. Quastel.

http://www.newton.ac.uk/event/srqw03

Gran Sasso Quantum Meeting: from Many Particle Systems to Quantum Fluids

Nov. 28 - Dec. 1, 2018. GSSI, L' Aquila, Italy.
Organized by P. Antonelli, S. Cenatiempo, M. Correggi, P. Marcati.

https://indico.gssi.it/event/3/

Open positions

Postdoctoral Fellowship in Mathematical Physics

The Center of Advanced Applied Sciences of the Czech Technical University in Prague announces a postdoctoral position in mathematical physics in the research team "Geometry and spectral properties of quantum systems".

The position is for one year, available immediately, with the possibility of extension for a second year upon mutual agreement. The gross salary is approximately 45,000 CZK monthly before tax. There are no teaching duties associated with the position.

Applicants should have a PhD in mathematics or theoretical physics (or equivalent) obtained preferably after January 1, 2014. They must show strong research promise in at least one of the following research domains: Schrödinger operators; Jacobi and other structured matrices, orthogonal polynomials; spectral theory; partial differential equations; geometric analysis.

An experience in the project topic area is an advantage but not necessary. The applications including curriculum vitae (including list of publications), brief research statement (past, current and future interests) and two letters of recommendation should be sent by e-mail to Pavel Exner (exner@ujf.cas.cz) and Pavel Stovicek (stovicek@fjfi.cvut.cz). All documents should be submitted as pdf files. The letters of recommendation should be sent directly by the persons providing the reference.

Complete application packages should arrive before November 30, 2018; the application will be evaluated and the decision taken in the order of arrival. For any further information about the position please contact Pavel Exner and Pavel Stovicek on the e-mail addresses above.

Postdoctoral position in mathematical physics at H.I.T. (Israel)

The Faculty of Sciences at Holon Institute of Technology (Israel) invites applications for a postdoctoral position in mathematical physics funded by the Israel Science Foundation and the Holon Institute of Technology.

The successful candidate will be involved in several exciting research projects lying at the crossroads of random matrix theory, quantum chaology and the theory of integrable systems. Previous research record in random matrix theory and related fields is an advantage, strong analytical skills is a must.

The initial appointment will be for one year with a possibility of an extension for up to two more years depending on performance and available funds.

Qualified candidates with a PhD degree in mathematics, mathematical or theoretical physics are encouraged to submit a letter of motivation, curriculum vitae, list of publications, and contact information of three referees to Eugene Kanzieper (eugene.kanzieper@gmail.com). For further information please browse: eugenekanzieper.faculty.hit.ac.il.

The expected starting date is October-November 2018 (negotiable). Applications will be reviewed as received until the position is filled.

For more information on these positions and for an updated list of academic job announcements in mathematical physics and related fields visit

http://www.iamp.org/page.php?page=page_positions

Benjamin Schlein (IAMP Secretary)

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