Laudatio for Prof. Nalini Anantharaman

Prof. S. De Bièvre
Université Lille 1

August 6, 2012

It is a pleasure and an honour for me to participate in this prize ceremony by presenting the work of Nalini Anantharaman in a few words. The citation of the Jury for the 2012 Henri Poincaré Prize reads: "For her original contributions to the area of quantum chaos, dynamical systems and Schrödinger equations, including a remarkable advance in the problem of quantum unique ergodicity."

Let me try to sketch informally the remarkable advance in quantum chaos the jury is referring to. For that purpose, think of quantum chaos as a branch of semi-classical analysis. And of semi-classical analysis as the branch of mathematics that makes rigorous a body of knowledge – accumulated over centuries – explaining the behaviour of high-frequency waves in terms of that of an associated Hamiltonian system. In other words, semi-classical analysis links PDE's to dynamical systems. Its application to quantum mechanical systems that have a "chaotic" classical limit goes by the name of "quantum chaos". There are two central questions in this field. One pertains to the high energy behaviour of the eigenvalues of the Hamiltonian operators of such systems, the other to that of their eigenfunctions. It is to the second question Nalini made a groundbreaking contribution. Let me explain.

It has been known for several decades that the eigenfunctions of a classically *ergodic* system "typically" equidistribute on the energy shell in phase space. This is the content of the Schnirelman theorem proven in the seventies and eighties to various degrees of generality and in different contexts by Schnirelman, Zelditch, Colin-de-Verdiere, Helffer-Martinez-Robert. Two remarks need to be made in order to understand Nalini's essential contribution. First, note that the Schnirelman theorem does not require the classical dynamics to be "chaotic" in any real sense: ergodic will do. Second, one has to understand better what is meant by "typical". For that purpose, recall that one can associate to each eigenfunction of the quantum system its Wigner function, which is a natural distribution on phase space. These Wigner functions have a set of accumulation points – known as semi-classical measures – and it is well known that those are invariant measures of the classical Hamiltonian dynamics. The Schnirelman theorem says that there is a density one sequence of eigenfunctions whose Wigner functions converge to the Liouville measure: this is what is meant by equidistribution. But the theorem leaves open the possibility that other accumulation points might exist. This begs the question: "Is it possible to characterize the set of such accumulation points among all invariant measures? And how does this set depend on the properties of the underlying classical dynamics? In particular, under what conditions is the set of accumulation points reduced to a single point, the Liouville measure?" The system is then said to be uniquely quantum ergodic, a term coined by Rudnick and Sarnak, who conjectured it to hold for Laplace-Beltrami operators on negatively curved compact Riemannian manifolds. A strong result in that direction was proven by Lindenstrauss in 2006, but only for arithmetic surfaces .

Nalini Anantharaman addressed the above questions for Laplace-Beltrami operators on arbitrary compact Riemannian manifolds of (non-constant) negative curvature. The geodesic flow on such manifolds, which is Hamiltonian, is known to be Anosov, and hence chaotic in the usual sense of "sensitive dependence on initial conditions." In 2006, she proved a lower bound on the topological entropy of the support of the semi-classical measures. This bound restricts the set of invariant measures that can be semi-classical measures, in the sense defined above. As an example, let me point out that an easily stated and understood – but still spectacular – consequence of this result is that semi-classical measures cannot be supported on a finite number of periodic orbits of the classical flow since such a measure would have zero topological entropy. In later work she and her collaborators (H. Koch, Nonnenmacher, G. Rivière) obtained stronger results, proving in particular lower bounds on the metric entropy of the semi-classical measures. These bound are sharp in the sense that they have been shown to be saturated in related models where semi-classical measures saturating them have been constructed.

Why do these results generate such enthusiasm? In my view, because they are the first results since the Schnirelman theorem that provide information on the set of semi-classical measures using only dynamical properties of the classical flow beyond mere ergodicity – in casu its Anosov character – and no other special algebraic or arithmetic structures. In fact, the proofs are expected to be robust: they should apply, mutatis-mutandis, to other quantum systems having an Anosov Hamiltonian classical limit.

It is clear it takes someone with exceptional skill, with an excellent background in dynamical systems theory, with the courage, perseverance and capacity to master the intricacies of semi-classical analysis, and finally with bright and novel ideas, to make such a breakthrough contribution. Nalini Anantharaman is just such a person. She did her undergraduate studies at the École Normale in Paris. She then wrote a PhD in dynamical systems theory under the supervision of Prof. Ledrappier at the Université Paris 6, which she completed when she was twenty-four, in the year 2000. In the following years, she undertook the work I just sketched. She was nominated professor at the Universit d'Orsay in 2009. Nalini has obtained other interesting results in Schrödinger and wave equation theory, that I have no time to elaborate on here, and there is no doubt she will delight us with many more beautiful theorems in the coming years.

In the name of all, I would like to end by congratulating her heartily on the occasion of her winning the Henri Poincaré Prize 2012. And finally, since Nalini could not be with us to receive her prize, I will be glad to honour a request of the organizers: that is to take it home with me!